Many Sequential Iterative Algorithms Can Be Parallel and (Nearly) Work-efficient

Zheqi Shen, Zijin Wan, Yan Gu, Yihan Sun

UC Riverside
Sequential Iterative Algorithms

Process objects one by one in order

for i = 1 to n
  process object i
Parallelizing Sequential Iterative Algorithms

Identify the **dependences** among objects

```
for i = 1 to n
    process object i
```
Parallelizing Sequential Iterative Algorithms

Obtain *parallelism*
Many Sequential Iterative Algorithms Can Be Parallel

Ideal Parallel Algorithm
- Parallelize vertices as much as possible
- Process a vertex only when it is ready

dependence graph (DG)

1  2
3  6
4  5

G. Blelloch et al. (PPoPP 2012)
G. Blelloch et al. (SPAA 2012)
W. Hasenplaugh et al. (SPAA 2014)
X. Pan et al. (NIPS 2015)
J. Shun et al. (SODA 2015)
M. Fischer and A. Noever. (SODA 2018)
G. Blelloch et al. (JACM 2020)
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...
Ideal Parallel Algorithm

- Round-efficiency - Parallelize vertices as much as possible
- Work-efficiency - Process a vertex only when it is ready
Efficiently Parallelizing Algorithms

Analysis is based on the **binary-forking model** (with TAS)
- **Work**: total number of operations
- **Span**: length of the longest execution path

Keep the algorithms efficient
- **Round-efficiency**
- **Work-efficiency**
Efficiently Parallelizing Algorithms

Analysis is based on the **binary-forking model** (with TAS)
- **Work**: total number of operations
- **Span**: length of the longest execution path

Keep the algorithms efficient
- **Round-efficiency**: $\tilde{O}(D)$ span ($D$=longest path length of the given DG)
- **Work-efficiency**:
Analysis is based on the **binary-forking model** (with TAS)
- **Work**: total number of operations
- **Span**: length of the longest execution path

Keep the algorithms efficient

- **Round-efficiency**: not equivalent to optimal span
- **Work-efficiency**: important for practical performance
- **Near work-efficiency**:

Some algorithms could not be parallelized efficiently
Example: Longest Increasing Subsequence (LIS)

Given a sequence $s_1 \ldots s_n$

The LIS problem finds the longest subsequence $s^*$ of $s$

Elements in $s^*$ are strictly increasing
Example: Longest Increasing Subsequence (LIS)

\[
dp[i] = \max_{j < i, s_j < s_i} dp[j] + 1
\]

Sequentially we can compute this in \(O(n \log n)\) cost.
Example: Longest Increasing Subsequence (LIS)

Existing parallel algorithms are not nearly work-efficient or round efficient
- Galil et al. (Parallel Distrib 1994), Krusche, et al. (PPAM 2009), Nakashima et al. (ISPDC 2002), Semé, Thierry, et al. (ICCSA 2006), Thierry, et. al. (SPAA 2001) have $\Omega(n^{1.5})$ work
- Alam and Rahman’s algorithm (IPL 2013) has $\Theta(n)$ span
- Krusche and Tiskin’s algorithm (SPAA 2010) has $\tilde{\Theta}(n \log^2 n)$ work and $\tilde{O}(n^{2/3})$ span
Example: Longest Increasing Subsequence (LIS)

General approaches / frameworks
- Do not directly give efficient solutions to LIS
Deterministic Reservations

Proposed by Blelloch et al. (PPoPP 2012), used in algorithms from (Shun et al., SODA 2015)

In a high level:

- Access all unprocessed objects each round
Deterministic Reservations

Proposed by Blelloch et al. (PPoPP 2012), used in algorithms from (Shun et al., SODA 2015)

In a high level:
- Access all unprocessed objects each round
- Check their readiness
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- Process the ready objects
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In a high level:
- Access all unprocessed objects each round
- Check their readiness
- Process the ready objects

- Work-efficient only when the work decreases geometrically in every round
- $\Theta(\text{round} \cdot n)$ work for LIS
- $O(n^2)$ worst-case
Activation-based Approaches

Used in Blelloch et al. (SPAA 2012, SPAA 2020), M. Fischer and A. Noever (SODA 2018), generalized by Blelloch et al. (SPAA 2020)

- Activate some successors based on the edges in the DG
Activation-based Approaches

- Activate some successors based on the edges in the DG
- Process the ready ones

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Activation-based Approaches

Used in Blelloch et al. (SPAA 2012, SPAA 2020), M. Fischer and A. Noever (SODA 2018), generalized by Blelloch et al. (SPAA 2020)

- Activate some successors based on the edges in the DG
- Process the ready ones
- Go through all the edges
- Take $\Theta(m)$ work for LIS ($m = \#\text{edges in the DG}$)
- $m$ can be up to $\Theta(n^2)$
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<th># candidates in next round</th>
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This Work

Phase-parallel framework to analyze dependences
- Core concept: rank
- Vertex-centric manner: avoid checking all the edges

Two general techniques to design algorithms in this framework
- Type 1 (more interesting problem in our paper)
- Type 2 (LIS problem is here)
Phase-parallel Framework

A general technique to parallelize the iterative algorithms

Formally define **rank** from the independence system \((S, \mathcal{F})\)

- Feasible set: \( \mathcal{F}(x) = \{E \in \mathcal{F}: E \subseteq x^\perp, x \in E\} \)
- Maximum feasible set: \( \text{MFS}(x) = \arg \max_{E \in \mathcal{F}(x)} |E| \)
- Rank: \( \text{rank}(x) = |\text{MFS}(x)| \)
Apply to LIS

\[ \mathcal{F}(x) = \{ E \in \mathcal{F} : E \subseteq x^\perp, x \in E \} \]
\[ \text{MFS}(x) = \arg \max_{E \in \mathcal{F}(x)} |E| \]
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Apply to LIS

Given an object $x$

- $\mathcal{F}(x) = \{E \in \mathcal{F}: E \subseteq x^\perp, x \in E\}$
- $\text{MFS}(x) = \arg \max_{E \in \mathcal{F}(x)} |E|$
- $\text{rank}(x) = |\text{MFS}(x)|$
Apply to LIS

Given an object $x$

- $\mathcal{F}(x)$: increasing subsequence ending with object $x$
- $\text{MFS}(x) = \arg \max_{E \in \mathcal{F}(x)} |E|$
- $\text{rank}(x) = |\text{MFS}(x)|$
Apply to LIS

Given an object $x$

- $\mathcal{F}(x)$: increasing subsequence ending with object $x$
- $\text{MFS}(x)$: LIS ending with object $x$
- $\text{rank}(x) = |\text{MFS}(x)|$
Apply to LIS

Given an object $x$

- $\mathcal{F}(x)$
  - increasing subsequence ending with object $x$
- $\text{MFS}(x)$
  - LIS ending with object $x$
- $\text{rank}(x)$
  - the length of LIS

$\text{rank}(9) = 3$
Apply to LIS

Given an object $x$

- $\mathcal{F}(x)$: increasing subsequence ending with object $x$
- MFS($x$): LIS ending with object $x$
- rank($x$): the length of LIS

The earliest “phase” when an object is ready

With certain conditions, the rank of an object is its depth in DG
Processes objects in the order of ranks
Processes objects in the order of ranks

Challenge
Find the ready objects in each round
Avoid visiting all edges

Solution
“Vertex-centric” approach
Core idea: select a “pivot” for every object
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Pivot: an unfinished predecessor selected at uniformly random

If the pivot of an object hasn’t finished, this object cannot be ready

We check readiness of an object only when its pivot finishes
Core idea: select a “pivot” for every object

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Pivot: an unfinished predecessor selected uniformly at random

If the pivot of an object haven’t finished, this object cannot be ready

We check readiness of an object only when its pivot finishes
Wake-up and re-pivoting

If the object waked up is not ready
- Update pivot to another unprocessed objects
- Sleep until being waked up the next time
Wake-up and re-pivoting

Assign a **pivot** to each object
Choose another pivot if not ready when being waked up

Save work as only a few edges are evaluated

If choosing pivots at uniformly random, 
#evaluated edges is $O(n \log n)$ w.h.p.
Apply to the LIS algorithm

F = \{\text{ready at the beginning}\}

\text{while } F \neq \emptyset \text{ \{ \\
\quad \text{process } F \\
\quad S_{woken} = \{\text{objects woken by } F\} \\
\quad S_{\text{ready}} = \{u \in S_{woken}, \ u \text{ is ready}\} \\
\quad \text{update pivots in } S_{woken} - S_{\text{ready}} \\
\quad F = S_{\text{ready}}
\}\}
Apply to the LIS algorithm

\[ F = \{ \text{ready at the beginning} \} \]

while \( F \neq \emptyset \)
{
    process \( F \)

    \( S_{\text{woken}} = \{ \text{objects woken by } F \} \)
    \( S_{\text{ready}} = \{ u \in S_{\text{woken}}, \ u \text{ is ready} \} \)

    update pivots in \( S_{\text{woken}} - S_{\text{ready}} \)
    \( F = S_{\text{ready}} \)
Apply to the LIS algorithm

\[ F = \text{\{ready at the beginning\}} \]

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Apply to the LIS algorithm

\[ F = \{ \text{ready at the beginning} \} \]

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Apply to the LIS algorithm

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while \( F \neq \emptyset \)

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\begin{align*}
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\end{align*}
\]
Apply to the LIS algorithm

F = \{\text{ready at the beginning}\}

while F ≠ ∅
{
    process F

    S_{\text{woken}} = \{\text{objects woken by F}\}
    S_{\text{ready}} = \{u ∈ S_{\text{woken}}, u \text{ is ready}\}

    \text{update pivots in } S_{\text{woken}} − S_{\text{ready}}

    F = S_{\text{ready}}
}

Apply to the LIS algorithm

\[ F = \{\text{ready at the beginning}\} \]

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Apply to the LIS algorithm

\[ F = \{ \text{ready at the beginning} \} \]

while \( F \neq \emptyset \) do

\[ \text{process } F \]

\[ S_{\text{woken}} = \{ \text{objects woken by } F \} \]

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\[ F = S_{\text{ready}} \]

\end{algorithm}
Vertex-centric Approach

\[ dp[i] = \max_{j<i, s_j<s_i} dp[j] + 1 \]
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Readiness check
DP value computation
Pivot re-selection

Above operations can be done by a 2D range tree in \( O(\log^2 n) \)

[Sun, et al. (PPoPP 2018)]
## Complexity of our parallel LIS algorithm

<table>
<thead>
<tr>
<th>input</th>
<th>( n )</th>
</tr>
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<tbody>
<tr>
<td># wake-ups</td>
<td>( O(n \log n) ) total wake-ups w.h.p.</td>
</tr>
<tr>
<td>per wake-up</td>
<td>( O(\log^2 n) ) by range queries</td>
</tr>
<tr>
<td>- Compute DP values</td>
<td></td>
</tr>
<tr>
<td>- Check readiness</td>
<td></td>
</tr>
<tr>
<td>- Re-select pivots</td>
<td></td>
</tr>
</tbody>
</table>
Parallel LIS Algorithm

Using our phase-parallel framework and vertex-centric methods, we parallelize LIS algorithm with

Nearly work-efficient: $O(n \log^3 n)$ work w.h.p.
Round-efficient: $O(r \log^2 n)$ span

$n = \text{input size}$
$r = \text{the LIS length of the input}$
Key Techniques

Pivots
- Wake-up strategy
- Vertex-centric readiness check

More applications in **Type-2** approach
- Other range-search-based greedy or dynamic programming
- Maximal independent set (MIS) using wake-up strategy only
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<th>Type-2 approach</th>
<th>Cost of single readiness check</th>
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<tr>
<td>Type-1 approach</td>
<td>Trivial</td>
<td>Logarithmic (by 1-D range queries)</td>
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# Improved Bounds

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<tr>
<td><strong>Activity Selection</strong> (weighted)</td>
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* with high probability

Huffman tree, graph coloring, SSSP, unlimited knapsack, and more in the paper
Experiment Setup

**Hardware**
- 96 CPU cores (192 hyper-threads)
- 1.5 TiB of main memory

**Parallelized Algorithms**
- Huffman Tree
- Activity selection
- LIS
  - ...

*Work-efficient algorithms generally perform well*
Huffman Tree

Our algorithm is work-efficient \( \tilde{O}(r) \) span for rank \( r \) (tree height)

The rank of the test data is low

The algorithm performs well
- 21.26x speedup to the sequential
Activity Selection

\[ n = 10^9 \]

Implement with both Type-1 and -2
- Both are work-efficient: \( O(n \log n) \)
- Span is \( \tilde{O}(r) \) for rank \( r \)

The algorithm performs well on a wide range of rank values

\( \approx 400k \)
Longest Increasing Subsequence

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<tr>
<th>Time (sec.)</th>
<th>Output size</th>
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<td>Parallel</td>
<td>Classic seq</td>
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- Nearly work-efficient: $O(n \log^3 n)$
- $O(\log^2 n)$ overhead

Run fast on small ranks
- Overhead in work still limits its performance for large ranks

Open problem:
$O(n \log n)$ work with good parallelism?
Summary

Motivation
- Many existing sequential iterative algorithms can be highly parallel
- Parallelize these algorithms efficiently

New Techniques: general to many algorithms
- Phase-parallel framework
- Type-1 and Type-2 approaches

New algorithms (many improving best known bounds)
- LIS, activity selection, MIS, Huffman tree, SSSP, …
Future Work

Can LIS problem be solved in $O(n \log n)$ with good parallelism?

Can our techniques apply to other problems?
Thank you
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