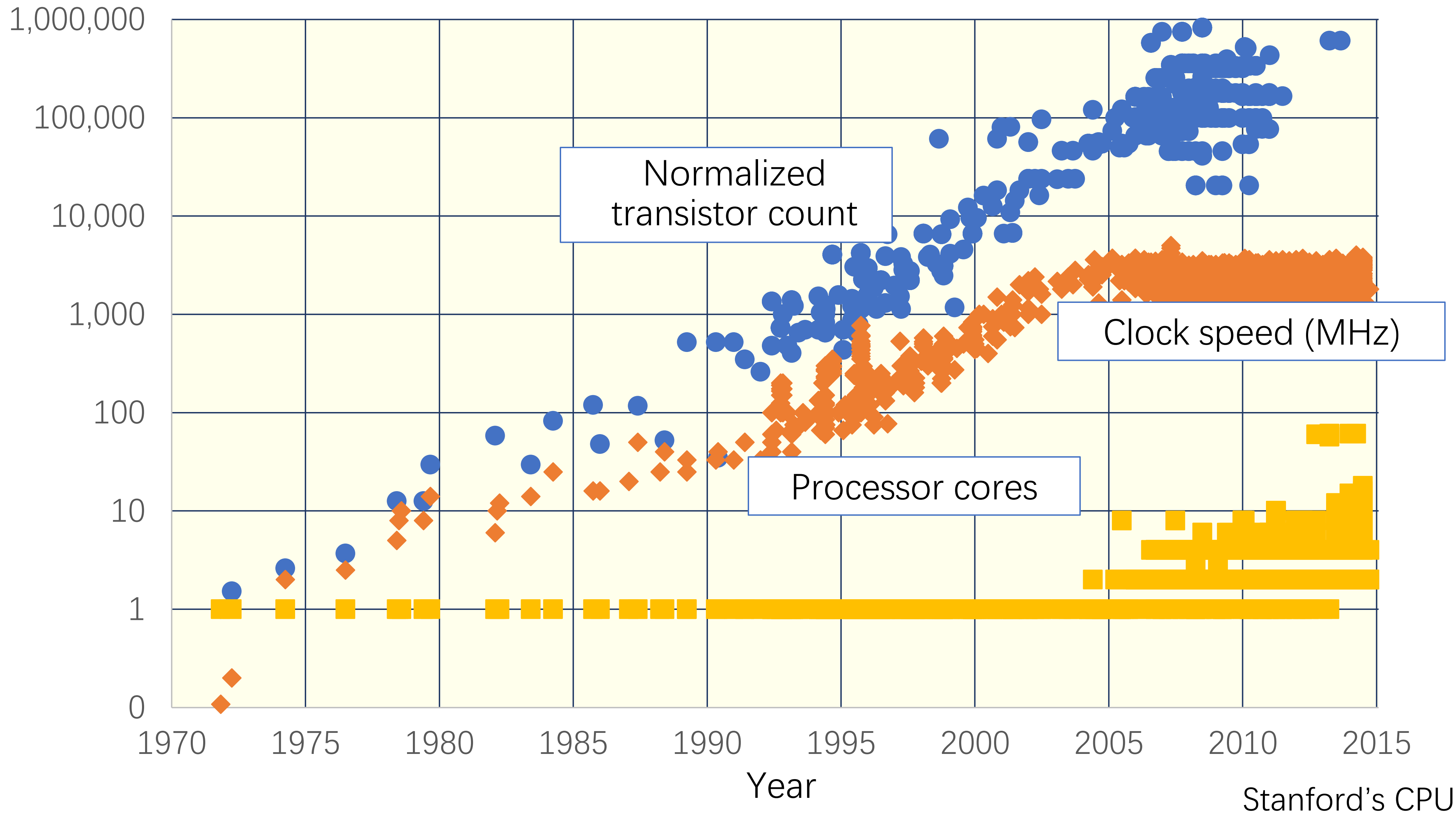
Algorithm Engineering
(aka. How to Write Fast Code)

CS260 – Lecture 2
Yan Gu

Case Study:
Matrix Multiplication

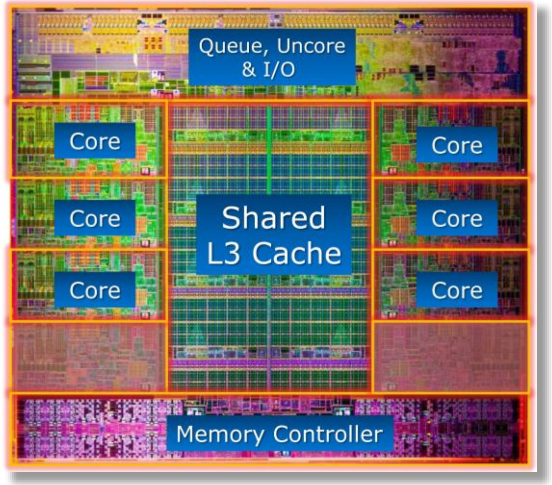
Many slides in this lecture are borrowed from the first lecture in 6.172 Performance Engineering of Software Systems at MIT. The credit is to Prof. Charles E. Leiserson, and the instructor appreciates the permission to use them in this course. The numbers of runtime and more details of the experiment can be found in Tao Schardl's dissertation *Performance engineering of multicore software: Developing a science of fast code for the post-Moore era*.

Technology Scaling

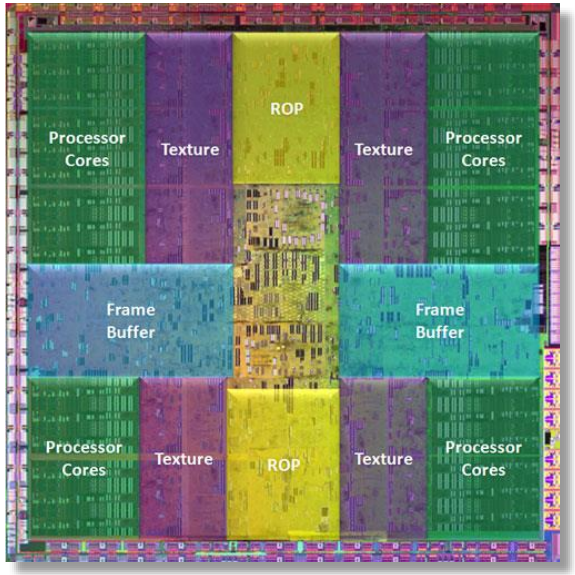


Stanford's CPU DB [DKM12]

Performance Is No Longer Free



2011 Intel Skylake processor



2008 NVIDIA GT200 GPU

- **Moore's Law continues to increase computer performance**
- **But now that performance looks like big multicore processors with complex cache hierarchies, wide vector units, GPUs, FPGAs, etc.**
- **Generally, algorithms must be adapted to utilize this hardware efficiently!**

Square-Matrix Multiplication

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$\underset{C}{} \qquad \underset{A}{} \qquad \underset{B}{}$$

C A B

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Assume for simplicity that $n = 2^k$.

AWS c4.8xlarge Machine Specs

Feature	Specification
Microarchitecture	Haswell (Intel Xeon E5-2666 v3)
Clock frequency	2.9 G
Processor chips	2
Processing cores	9 per processor chip
Hyperthreading	2 way
Floating-point unit	8 do fuse
Cache-line size	64 B
L1-icache	32 KB private 8-way set associative
L1-dcache	32 KB
L2-cache	256
L3-cache (LLC)	25 M
DRAM	60 GB

c4.8xlarge	$1.591 per Hour	$0.27 per Hour

c5.18xlarge	$3.06 per Hour	$0.27 per Hour
c5.24xlarge	$4.08 per Hour	$0.27 per Hour

Model	vCPU	Memory (GiB)
c5.18xlarge	72	144
c5.24xlarge	96	192

$$\text{Peak} = (2.9 \times 10^9) \times 2 \times 9 \times 16 = 836 \text{ GFLOPS}$$

Version 1: Nested Loops in Python

```
import sys, random
from time import *

n = 4096

A = [[random.random()
      for row in xrange(n)]
     for col in xrange(n)]
B = [[random.random()
      for row in xrange(n)]
     for col in xrange(n)]
C = [[0 for row in xrange(n)]
     for col in xrange(n)]

start = time()
for i in xrange(n):
    for j in xrange(n):
        for k in xrange(n):
            C[i][j] += A[i][k] * B[k][j]
end = time()

print '%0.6f' % (end - start)
```

Running time = 21042 seconds ≈ 6 hours

Is this fast?

Should we expect more?

Version 1: Nested Loops in Python

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from time import *

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     for col in xrange(n)]
C = [[0 for row in xrange(n)]
     for col in xrange(n)]

start = time()
for i in xrange(n):
    for j in xrange(n):
        for k in xrange(n):
            C[i][j] += A[i][k]
end = time()

print '%0.6f' % (end - start)
```

Running time = 21042 seconds ≈ 6 hours

Is this fast?

Should we expect more?

Back-of-the-envelope calculation

$2n^3 = 2(2^{12})^3 = 2^{37}$ floating-point operations

Running time = 21042 seconds

∴ Python gets $2^{37}/21042 \approx 6.25$ MFLOPS

Peak ≈ 836 GFLOPS

Python gets ≈ 0.00075% of peak

Version 2: Java

```java
import java.util.Random;

public class mm_java {
  static int n = 4096;
  static double[][] A = new double[n][n];
  static double[][] B = new double[n][n];
  static double[][] C = new double[n][n];

  public static void main(String[] args) {
    Random r = new Random();

    for (int i=0; i<n; i++) {
      for (int j=0; j<n; j++) {
        A[i][j] = r.nextDouble();
        B[i][j] = r.nextDouble();
        C[i][j] = 0;
      }
    }

    long start = System.nanoTime();

    for (int i=0; i<n; i++) {
      for (int j=0; j<n; j++) {
        for (int k=0; k<n; k++) {
          C[i][j] += A[i][k] * B[k][j];
        }
      }
    }

    long stop = System.nanoTime();

    double tdiff = (stop - start) * 1e-9;
    System.out.println(tdiff);
  }
}
```

Running time = 2,738 seconds ≈ 46 minutes

… about 8.8× faster than Python.

```java
for (int i=0; i<n; i++) {
  for (int j=0; j<n; j++) {
    for (int k=0; k<n; k++) {
      C[i][j] += A[i][k] * B[k][j];
    }
  }
}
```

Version 3: C

```c
#include <stdlib.h>
#include <stdio.h>
#include <sys/time.h>

#define n 4096
double A[n][n];
double B[n][n];
double C[n][n];

float tdiff(struct timeval *start,
            struct timeval *end) {
  return (end->tv_sec-start->tv_sec) +
    1e-6*(end->tv_usec-start->tv_usec);
}

int main(int argc, const char *argv[]) {
  for (int i = 0; i < n; ++i) {
    for (int j = 0; j < n; ++j) {
      A[i][j] = (double)rand() / (double)RAND_MAX;
      B[i][j] = (double)rand() / (double)RAND_MAX;
      C[i][j] = 0;
    }
  }

  struct timeval start, end;
  gettimeofday(&start, NULL);

  for (int i = 0; i < n; ++i) {
    for (int j = 0; j < n; ++j) {
      for (int k = 0; k < n; ++k) {
        C[i][j] += A[i][k] * B[k][j];
      }
    }
  }

  gettimeofday(&end, NULL);
  printf("%0.6f\n", tdiff(&start, &end));
  return 0;
}
```


```c
for (int i = 0; i < n; ++i) {
  for (int j = 0; j < n; ++j) {
    for (int k = 0; k < n; ++k) {
      C[i][j] += A[i][k] * B[k][j];
    }
  }
}
```


Using the Clang/LLVM 5.0 compiler

Running time = 1,156 seconds ≈ 19 minutes

About 2× faster than Java and
about 18× faster than Python

Where We Stand So Far

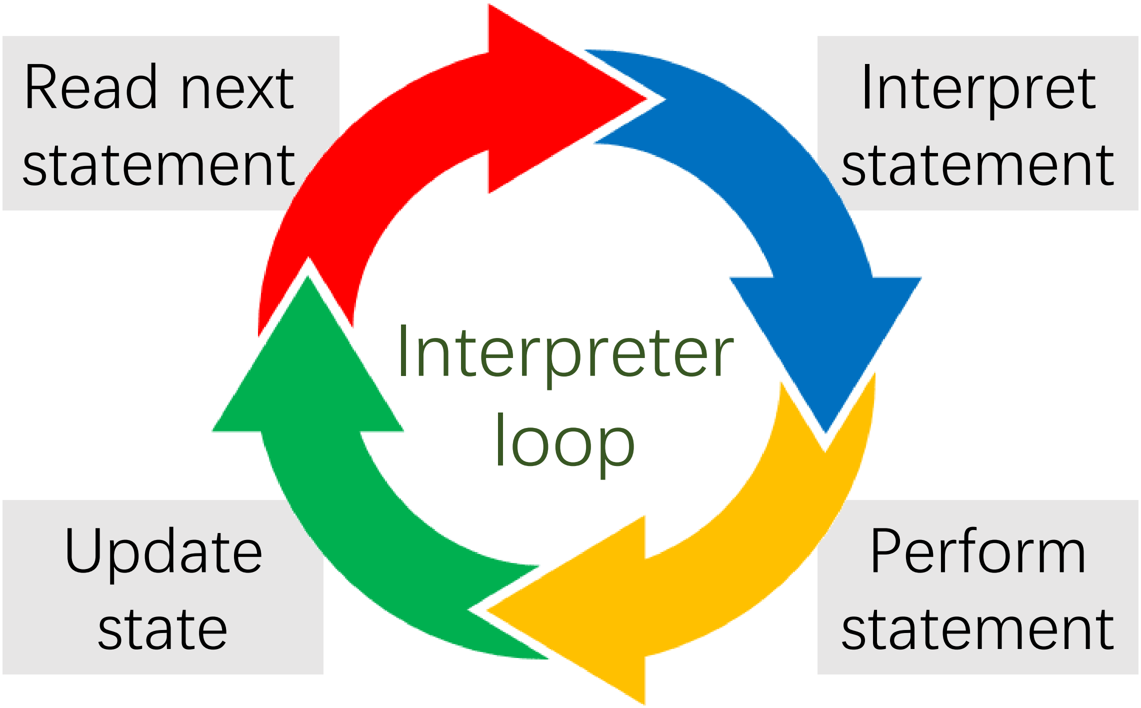
Version	Implementation	Running time (s)	Relative speedup	Absolute Speedup	GFLOPS	Percent of peak
1	Python	21041.67	1.00	1	0.007	0.001
2	Java	2387.32	8.81	9	0.058	0.007
3	C	1155.77	2.07	18	0.119	0.014

Why is Python so slow and C so fast?

- Python is interpreted
- C is compiled directly to machine code
- Java is compiled to byte-code, which is then interpreted and just-in-time (JIT) compiled to machine code

Interpreters are versatile, but slow

- **The interpreter reads, interprets, and performs each program statement and updates the machine state**
- **Interpreters can easily support high-level programming features — such as dynamic code alteration — at the cost of performance**



JIT Compilation

- JIT compilers can recover some of the performance lost by interpretation
- When code is first executed, it is interpreted
- The runtime system keeps track of how often the various pieces of code are executed
- Whenever some piece of code executes sufficiently frequently, it gets compiled to machine code in real time
- Future executions of that code use the more-efficient compiled version

Loop Order

We can change the order of the loops in this program without affecting its correctness

```
for (int i = 0; i < n; ++i) {
  for (int j = 0; j < n; ++j) {
    for (int k = 0; k < n; ++k) {
      C[i][j] += A[i][k] * B[k][j];
    }
  }
}
```

Loop Order

We can change the order of the loops in this program without affecting its correctness

```
for (int i = 0; i < n; ++i) {
    for (int k = 0; k < n; ++k) {
  for (int j = 0; j < n; ++j) {
      C[i][j] += A[i][k] * B[k][j];
    }
  }
}
```

Does the order of loops matter for performance?

Performance of Different Orders

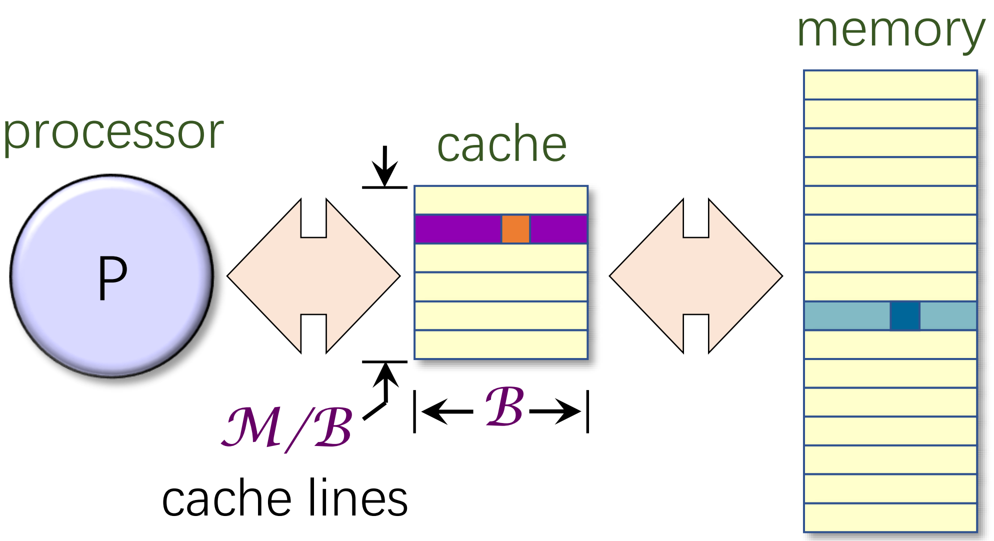
Loop order (outer to inner)	Running time (s)
i, j, k	1155.77
i, k, j	177.68
j, i, k	1080.61
j, k, i	3056.63
k, i, j	179.21
k, j, i	3032.82

- **Loop order affects running time by a factor of 18!**
- **What's going on?!**

Hardware Caches

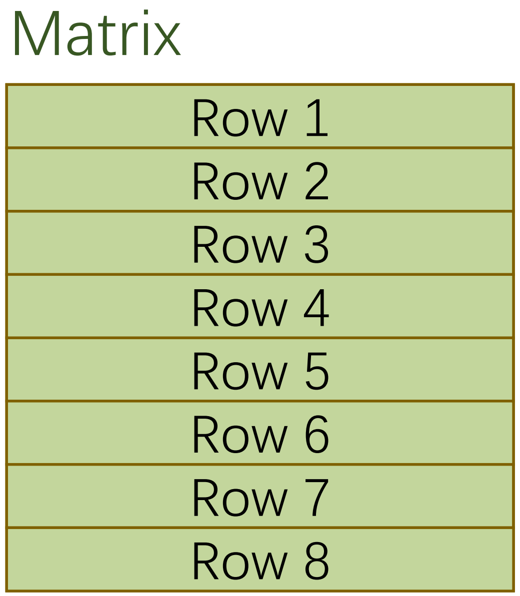
Each processor reads and writes main memory in contiguous blocks, called *cache lines*

- **Previously accessed cache lines are stored in a smaller memory, called a *cache*, that sits near the processor**
- ***Cache hits* — accesses to data in cache — are fast**
- ***Cache misses* — accesses to data not in cache — are slow**



Memory Layout of Matrices

In this matrix-multiplication code, matrices are laid out in memory in *row-major order*

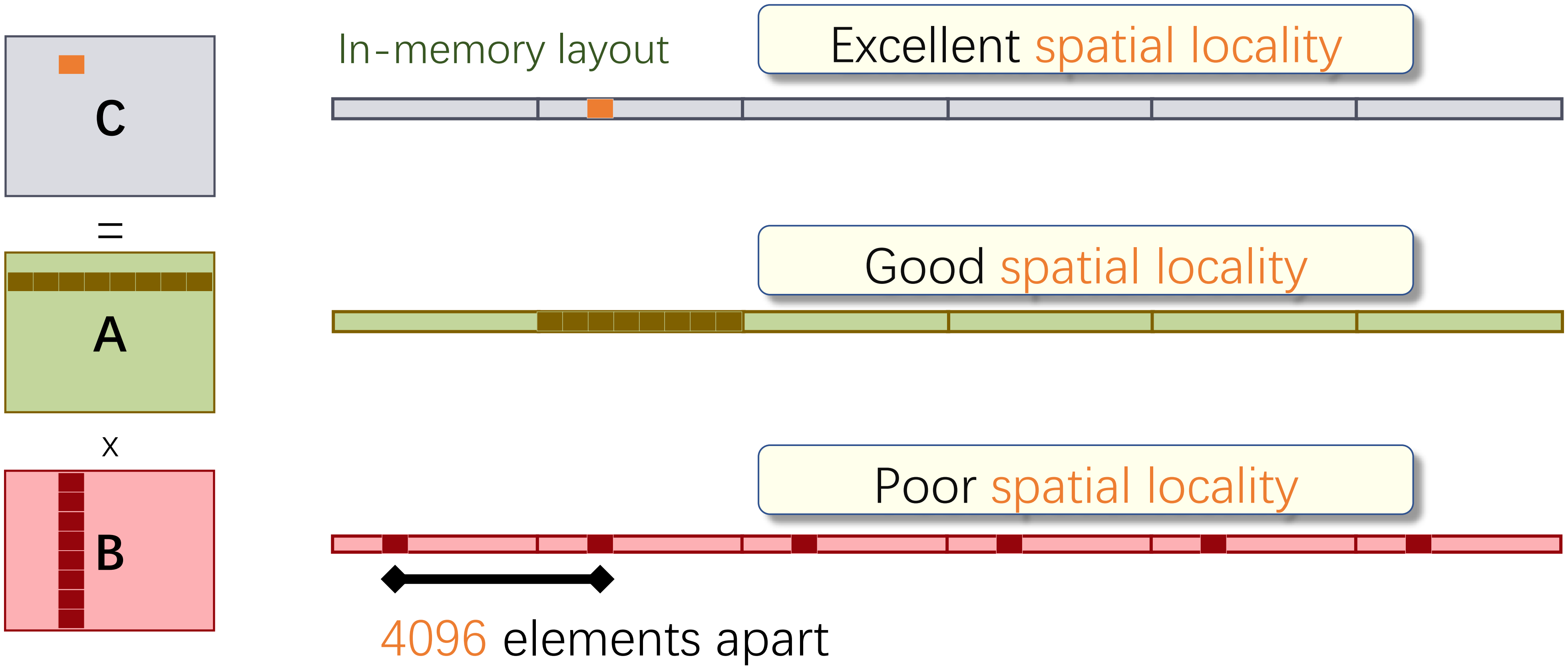


What does this layout imply about the performance of different loop orders?

Access Pattern for Order i, j, k

```
for (int i = 0; i < n; ++i)
  for (int j = 0; j < n; ++j)
    for (int k = 0; k < n; ++k)
      C[i][j] += A[i][k] * B[k][j];
```

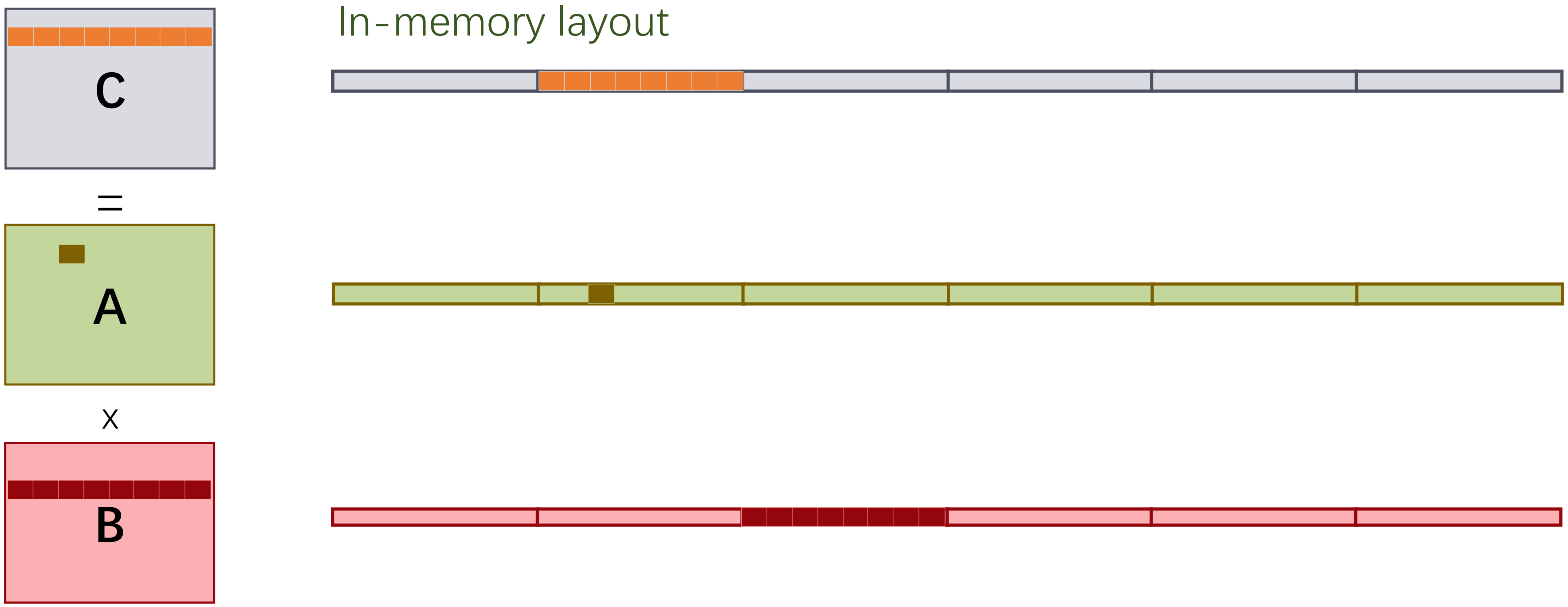
Running time:
1155.77s



Access Pattern for Order i, k, j

```
for (int i = 0; i < n; ++i)
  for (int k = 0; k < n; ++k)
    for (int j = 0; j < n; ++j)
      C[i][j] += A[i][k] * B[k][j];
```

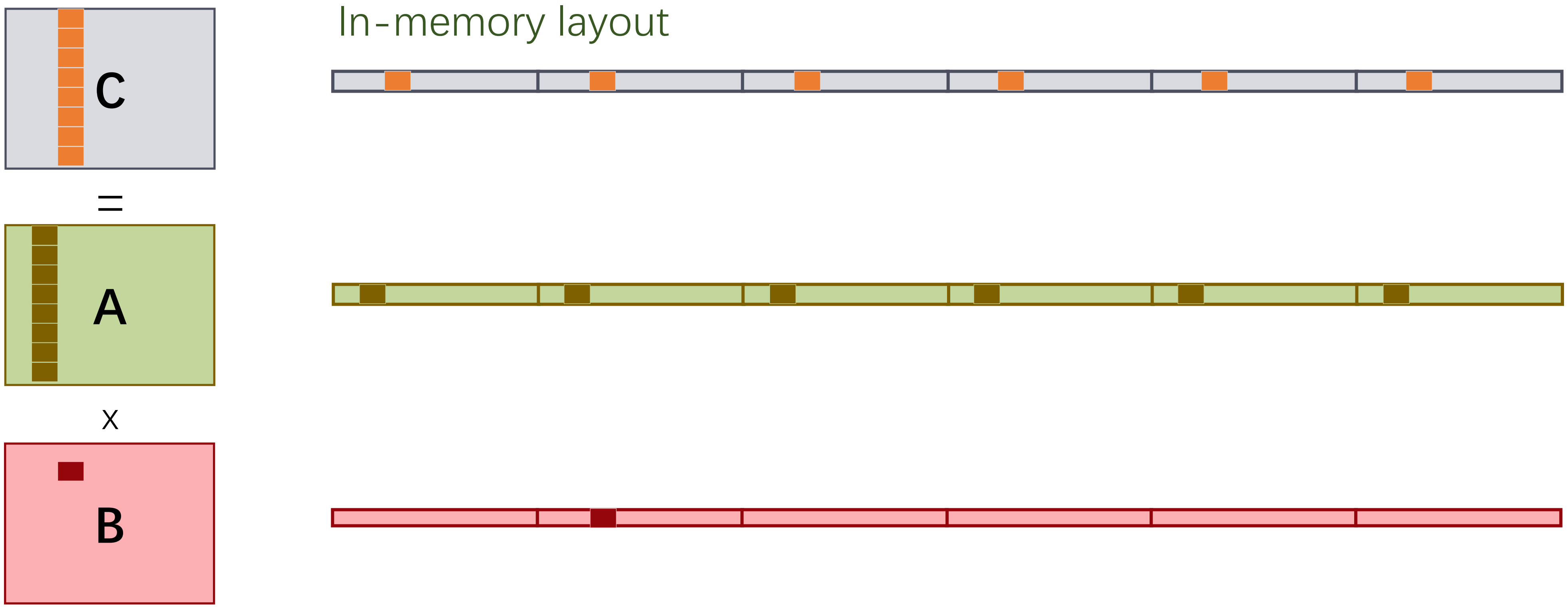
Running time:
177.68s



Access Pattern for Order j, k, i

```
for (int j = 0; j < n; ++j)
  for (int k = 0; k < n; ++k)
    for (int i = 0; i < n; ++i)
      C[i][j] += A[i][k] * B[k][j];
```

Running time:
3056.63s



Performance of Different Orders

We can measure the effect of different access patterns using the Cachegrind cache simulator:

```
$ valgrind --tool=cachegrind ./mm
```

Loop order (outer to inner)	Running time (s)	Last-level-cache miss rate
i, j, k	1155.77	7.7%
i, k, j	177.68	1.0%
j, i, k	1080.61	8.6%
j, k, i	3056.63	15.4%
k, i, j	179.21	1.0%
k, j, i	3032.82	15.4%

Version 4: Interchange Loops

Version	Implementation	Running time (s)	Relative speedup	Absolute Speedup	GFLOPS	Percent of peak
1	Python	21041.67	1.00	1	0.006	0.001
2	Java	2387.32	8.81	9	0.058	0.007
3	C	1155.77	2.07	18	0.118	0.014
4	+ interchange loops	177.68	6.50	118	0.774	0.093

What other simple changes we can try?

Compiler Optimization

Clang provides a collection of optimization switches. You can specify a switch to the compiler to ask it to optimize

Opt. level	Meaning	Time (s)
-O0	Do not optimize	177.54
-O1	Optimize	66.24
-O2	Optimize even more	54.63
-O3	Optimize yet more	55.58

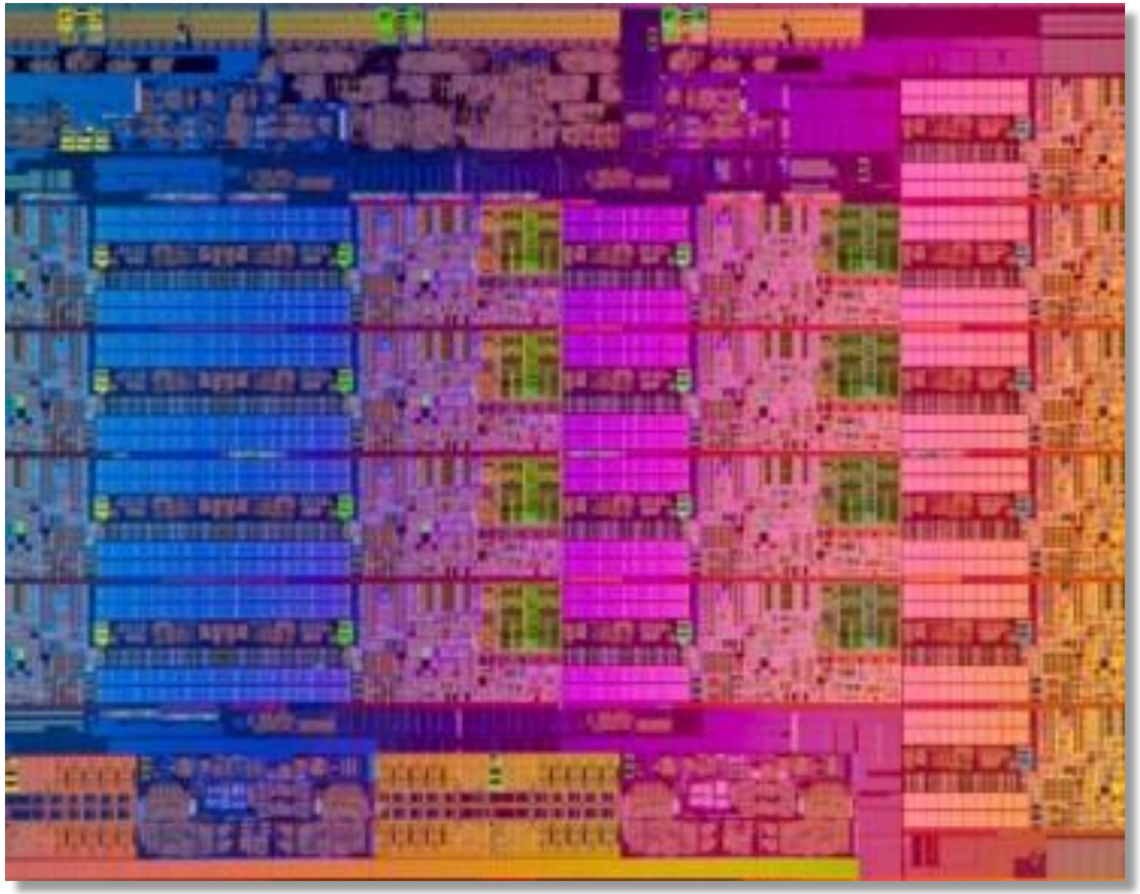
Version 5: Optimization Flags

Version	Implementation	Running time (s)	Relative speedup	Absolute Speedup	GFLOPS	Percent of peak
1	Python	21041.67	1.00	1	0.006	0.001
2	Java	2387.32	8.81	9	0.058	0.007
3	C	1155.77	2.07	18	0.118	0.014
4	+ interchange loops	177.68	6.50	118	0.774	0.093
5	+ optimization flags	54.63	3.25	385	2.516	0.301

With simple code and compiler technology, we can achieve 0.3% of the peak performance of the machine

What's causing the low performance?

Multicore Parallelism



Intel Haswell E5:
9 cores per chip

The AWS test machine has 2 of these chips

We're running on just 1 of the 18 parallel-processing cores on this system. Let's use them all!

Parallel Loops

The cilk_for loop allows all iterations of the loop to execute in parallel

```
cilk_for (int i = 0; i < n; ++i)
  for (int k = 0; k < n; ++k)
    cilk_for (int j = 0; j < n; ++j)
      C[i][j] += A[i][k] * B[k][j];
```

These loops can be (easily) parallelized.

Which parallel version works best?

Experimenting with Parallel Loops

Parallel `i` loop

```
cilk_for (int i = 0; i < n; ++i)
  for (int k = 0; k < n; ++k)
    for (int j = 0; j < n; ++j)
      C[i][j] += A[i][k] * B[k][j];
```

Running time: 3.18s

Parallel `j` loop

```
for (int i = 0; i < n; ++i)
  for (int k = 0
    cilk_for (in
      C[i][j] +=
```

Running time: 531.71s

Rule of Thumb

Parallelize outer loops rather than inner loops

Parallel `i` and `j`

```
cilk_for (int i = 0; i < n; ++i)
  for (int k = 0; k < n; ++k)
    cilk_for (int j = 0; j < n; ++j)
      C[i][j] += A[i][k] * B[k][j];
```

Running time: 10.64s

Version 6: Parallel Loops

Version	Implementation	Running time (s)	Relative speedup	Absolute Speedup	GFLOPS	Percent of peak
1	Python	21041.67	1.00	1	0.006	0.001
2	Java	2387.32	8.81	9	0.058	0.007
3	C	1155.77	2.07	18	0.118	0.014
4	+ interchange loops	177.68	6.50	118	0.774	0.093
5	+ optimization flags	54.63	3.25	385	2.516	0.301
6	Parallel loops	3.04	17.97	6,921	45.211	5.408

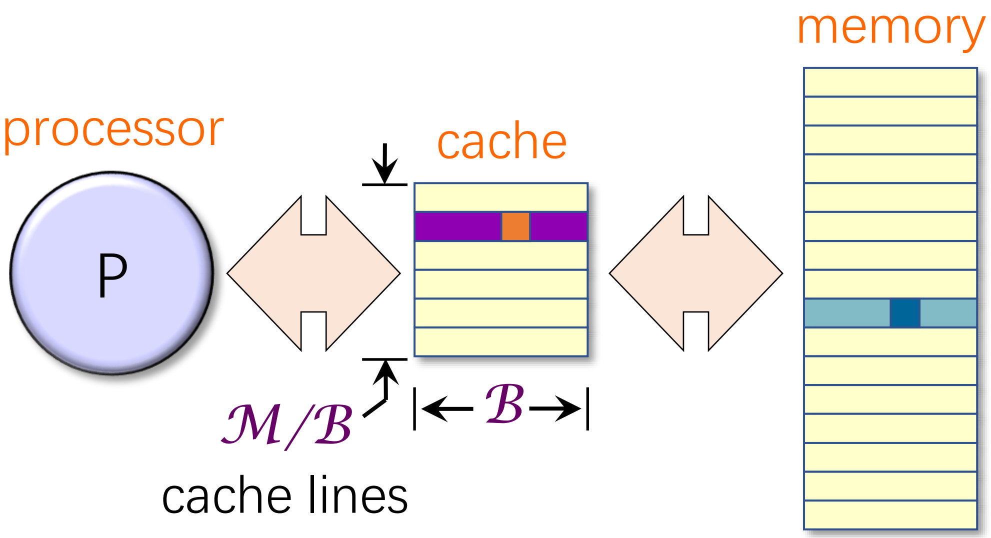
Using parallel loops gets us almost 18× speedup on 18 cores!
(Disclaimer: Not all code is so easy to parallelize effectively.)

Why are we still getting just 5% of peak?

Hardware Caches, Revisited

IDEA: **Restructure the computation to reuse data in the cache as much as possible**

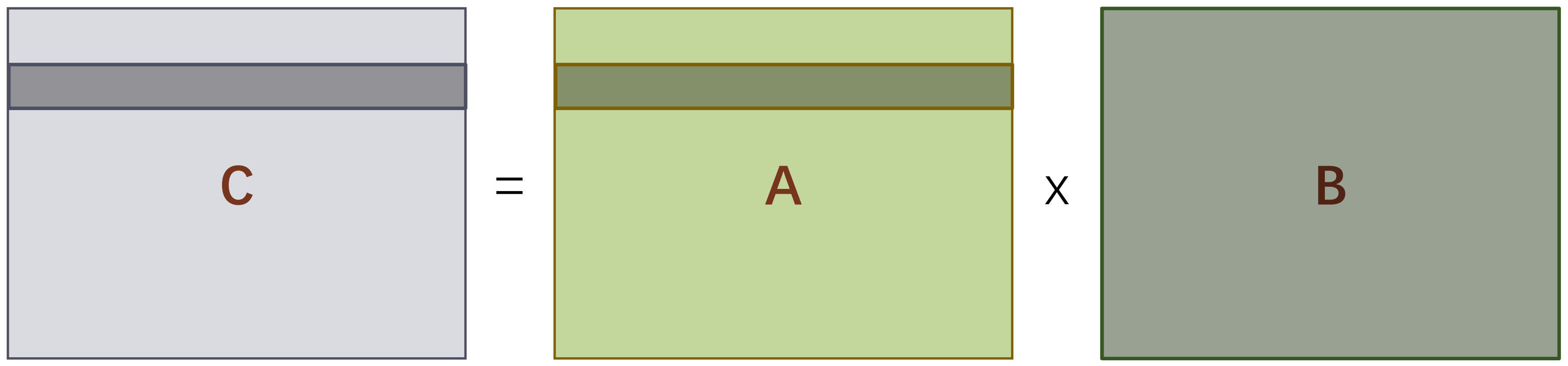
- **Cache misses are slow, and cache hits are fast**
- **Try to make the most of the cache by reusing the data that's already there**



Data Reuse: Loops

How many memory accesses must the looping code perform to fully compute 1 row of C?

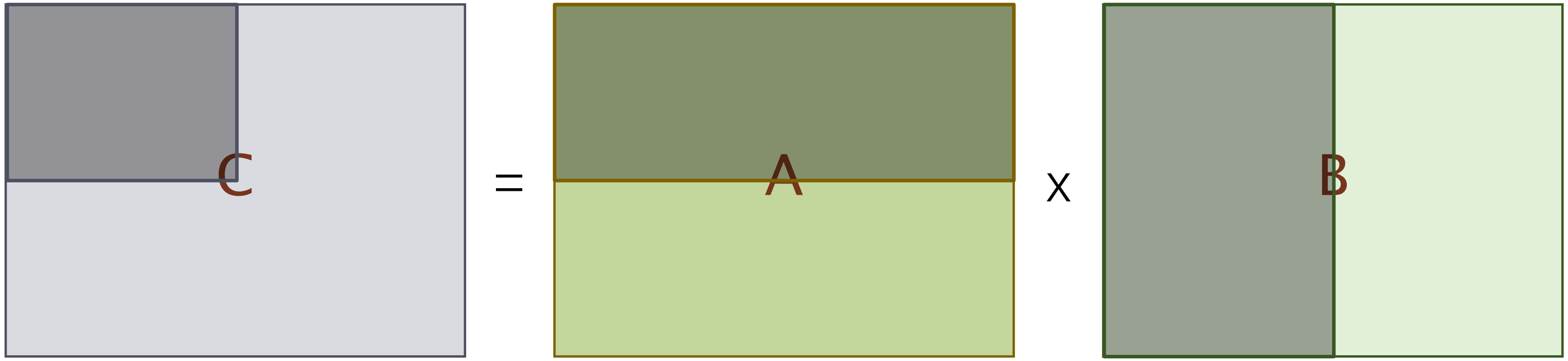
- **4096 * 1 = 4096 writes to C,**
- **4096 * 1 = 4096 reads from A, and**
- **4096 * 4096 = 16,777,216 reads from B, which is**
- **16,785,408 memory accesses total**



Data Reuse: Blocks

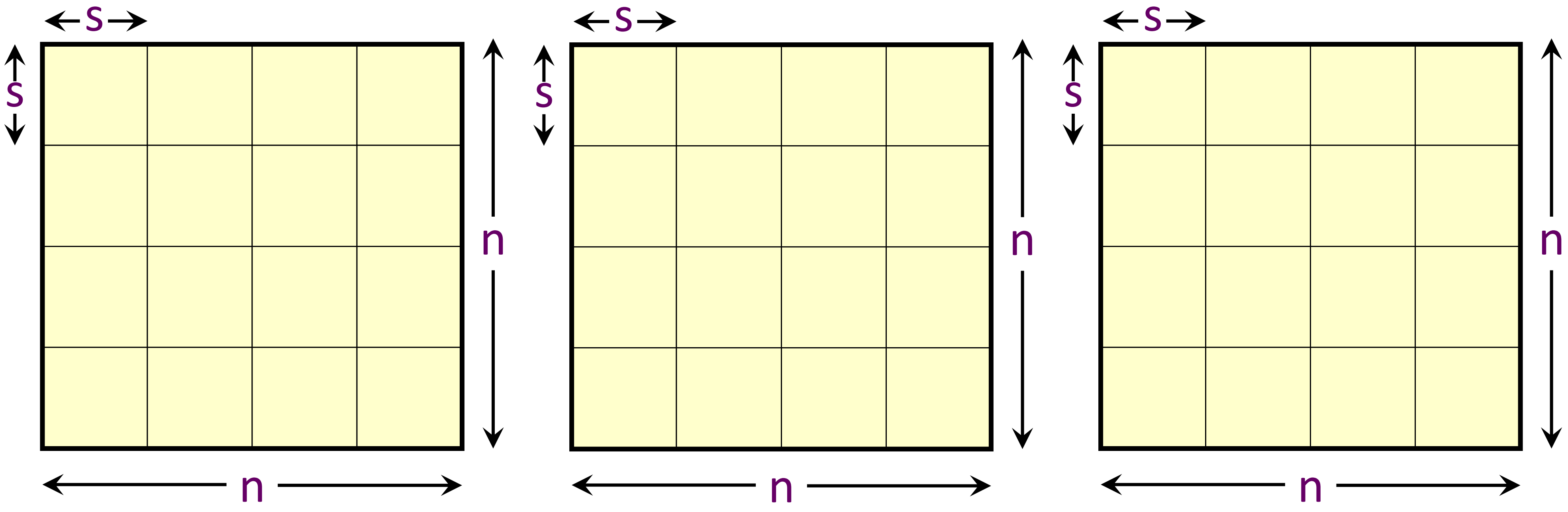
How about to compute a 64×64 block of C?

- **$64 \cdot 64 = 4096$ writes to C,**
- **$64 \cdot 4096 = 262{,}144$ reads from A, and**
- **$4096 \cdot 64 = 262{,}144$ reads from B, or**
- **528,384 memory accesses total**



Tiled Matrix Multiplication

```
cilk_for (int ih = 0; ih < n; ih += s)
  cilk_for (int jh = 0; jh < n; jh += s)
    for (int kh = 0; kh < n; kh += s)
      for (int il = 0; il < s; ++il)
        for (int kl = 0; kl < s; ++kl)
          for (int jl = 0; jl < s; ++jl)
            C[ih+il][jh+jl] += A[ih+il][kh+kl] * B[kh+kl][jh+jl];
```



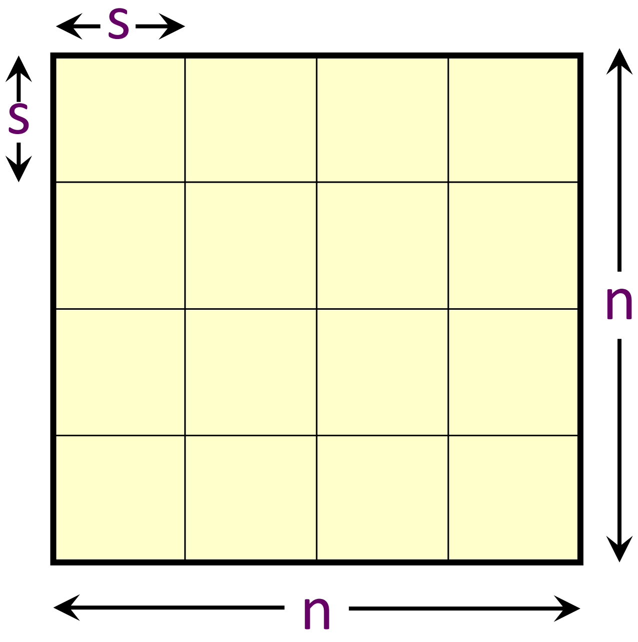
Tiled Matrix Multiplication

```
cilk_for (int ih = 0; ih < n; ih += s)
  cilk_for (int jh = 0; jh < n; jh += s)
    for (int kh = 0; kh < n; kh += s)
      for (int il = 0; il < s; ++il)
        for (int kl = 0; kl < s; ++kl)
          for (int jl = 0; jl < s; ++jl)
            C[ih+il][jh+jl] += A[ih+il][kh+kl] * B[kh+kl][jh+jl];
```

Tuning parameter

How do we find the right value of s?

Experiment!



Tile size	Running time (s)
4	6.74
8	2.76
16	2.49
32	1.74
64	2.33
128	2.13

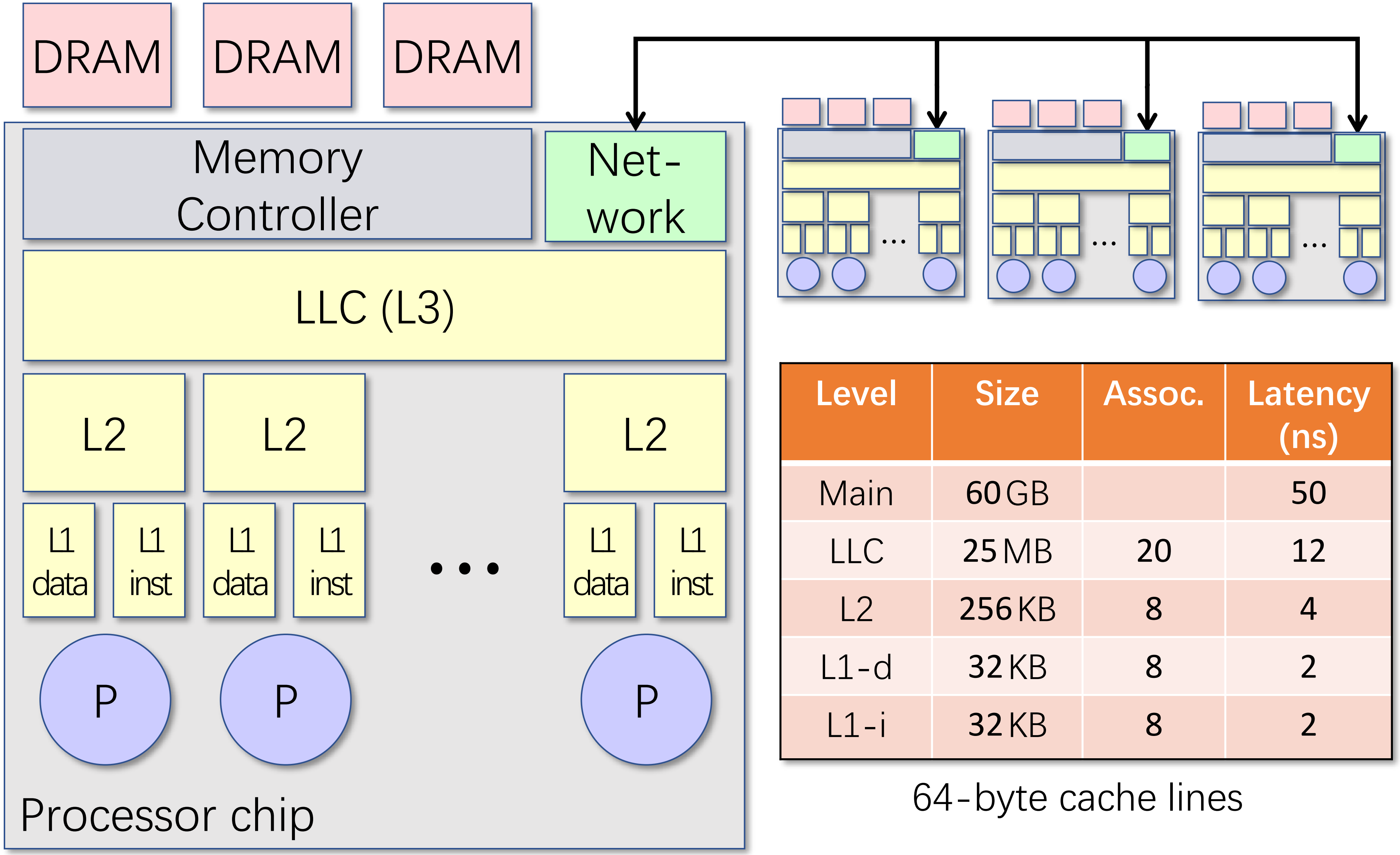
Version 7: Tiling

Version	Implementation	Running time (s)	Relative speedup	Absolute Speedup	GFLOPS	Percent of peak
1	Python	21041.67	1.00	1	0.006	0.001
2	Java	2387.32	8.81	9	0.058	0.007
3	C	1155.77	2.07	18	0.118	0.014
4	+ interchange loops	177.68	6.50	118	0.774	0.093
5	+ optimization flags	54.63	3.25	385	2.516	0.301
6	Parallel loops	3.04	17.97	6,921	45.211	5.408
7	+ tiling	1.74	1.70	11,772	76.782	9.184

Implementation	Cache references (millions)	L1-d cache misses (millions)	Last-level cache misses (millions)
Parallel loops	104,090	17,220	8,600
+ tiling	64,690	11,777	416

The tiled implementation performs about 62% fewer cache references and incurs 68% fewer cache misses.

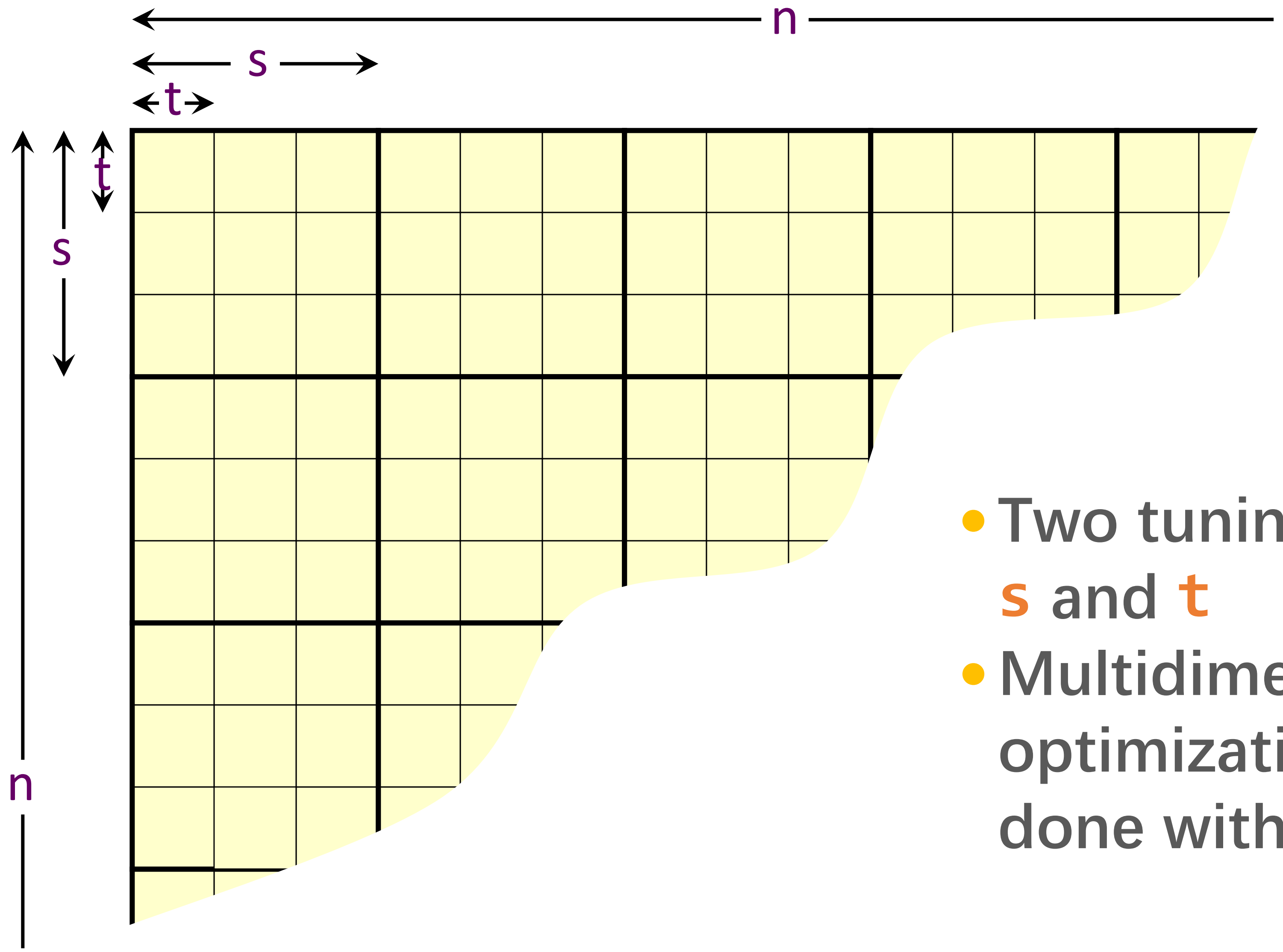
Multicore Cache Hierarchy



Level	Size	Assoc.	Latency (ns)
Main	60 GB		50
LLC	25 MB	20	12
L2	256 KB	8	4
L1-d	32 KB	8	2
L1-i	32 KB	8	2

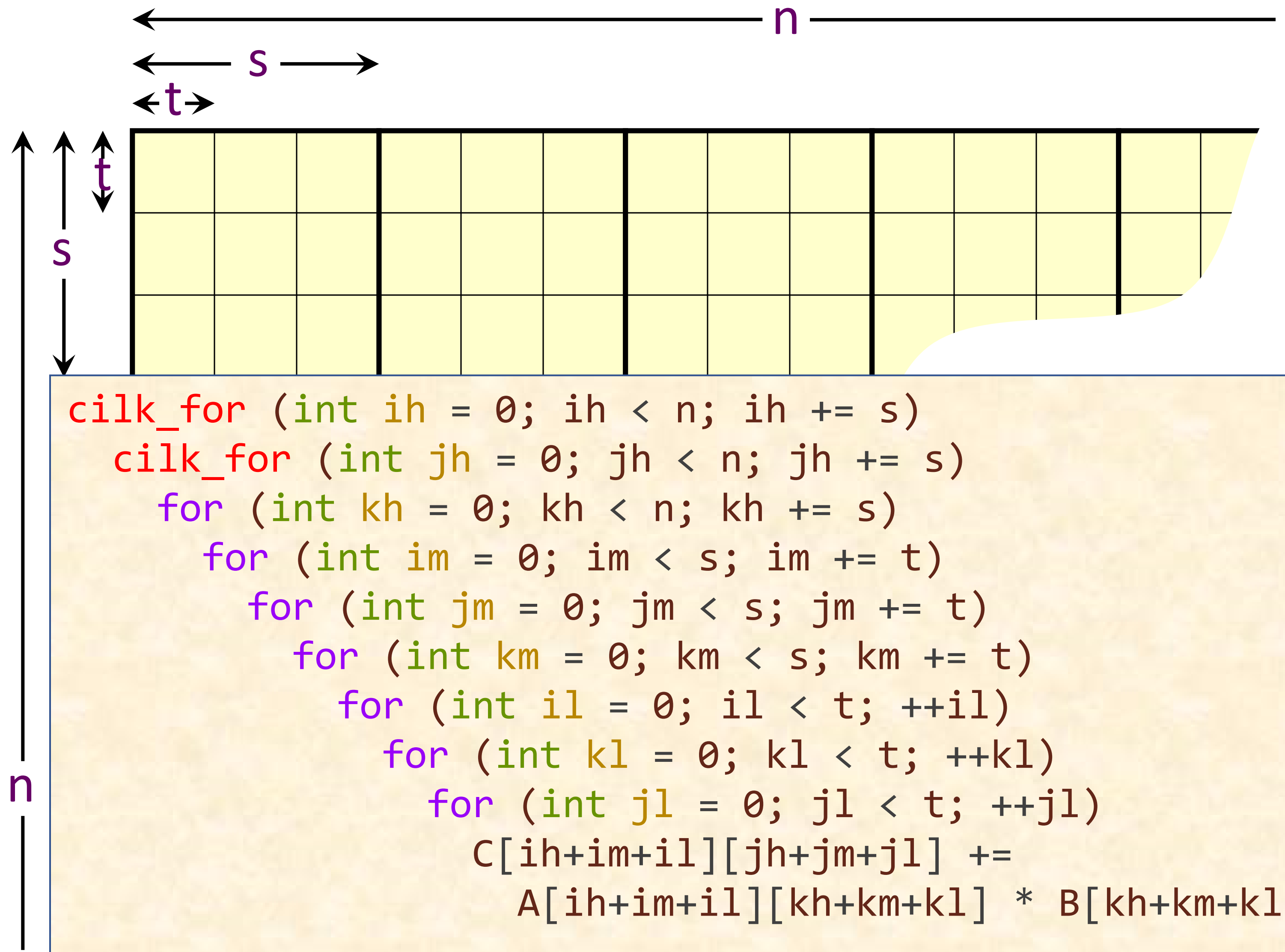
64-byte cache lines

Tiling for a Two-Level Cache



- Two tuning parameters, **s** and **t**
- Multidimensional tuning optimization cannot be done with binary search

Tiling for a Two-Level Cache




```
cilk_for (int ih = 0; ih < n; ih += s)
  cilk_for (int jh = 0; jh < n; jh += s)
    for (int kh = 0; kh < n; kh += s)
      for (int im = 0; im < s; im += t)
        for (int jm = 0; jm < s; jm += t)
          for (int km = 0; km < s; km += t)
            for (int il = 0; il < t; ++il)
              for (int kl = 0; kl < t; ++kl)
                for (int jl = 0; jl < t; ++jl)
                  C[ih+im+il][jh+jm+jl] +=
                    A[ih+im+il][kh+km+kl] * B[kh+km+kl][jh+jm+jl];
```

Recursive Matrix Multiplication

IDEA: Tile for every power of 2 simultaneously

$$\begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \cdot \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}$$

$$= \begin{bmatrix} A_{00}B_{00} & A_{00}B_{01} \\ A_{10}B_{00} & A_{10}B_{01} \end{bmatrix} + \begin{bmatrix} A_{01}B_{10} & A_{01}B_{11} \\ A_{11}B_{10} & A_{11}B_{11} \end{bmatrix}$$

8 multiplications of $n/2 \times n/2$ matrices
1 addition of $n \times n$ matrices

Recursive Parallel Matrix Multiply

The child function call is spawned, meaning it may execute in parallel with the parent caller

Control may not pass this point until all spawned children have returned.

```
void mm_dac(double *restrict C, int n_C,
            double *restrict A, int n_A,
            double *restrict B, int n_B,
            int n)
{ // C += A * B
  assert((n & (-n)) == n);
  if (n <= 1) {
    *C += *A * *B;
  } else {
#define X(M,r,c) (M + (r*(n_ ## M) + c)*(n/2))
    cilk_spawn mm_dac(X(C,0,0), n_C, X(A,0,0), n_A, X(B,0,0), n_B, n/2);
    cilk_spawn mm_dac(X(C,0,1), n_C, X(A,0,0), n_A, X(B,0,1), n_B, n/2);
    cilk_spawn mm_dac(X(C,1,0), n_C, X(A,1,0), n_A, X(B,0,0), n_B, n/2);
               mm_dac(X(C,1,1), n_C, X(A,1,0), n_A, X(B,0,1), n_B, n/2);
    cilk_sync;
    cilk_spawn mm_dac(X(C,0,0), n_C, X(A,0,1), n_A, X(B,1,0), n_B, n/2);
    cilk_spawn mm_dac(X(C,0,1), n_C, X(A,0,1), n_A, X(B,1,1), n_B, n/2);
    cilk_spawn mm_dac(X(C,1,0), n_C, X(A,1,1), n_A, X(B,1,0), n_B, n/2);
               mm_dac(X(C,1,1), n_C, X(A,1,1), n_A, X(B,1,1), n_B, n/2);
    cilk_sync;
  }
}
```

Recursive Parallel Matrix Multiply

```
void mm_dac(double *restrict C, int n_C,
            double *restrict A, int n_A,
            double *restrict B, int n_B,
            int n)
{ // C += A * B
  assert((n & (-n)) == n);
  if (n <= 1) {
    *C += *A * *B;
  } else {
#define X(M,r,c) (M + (r*(n_ ## M) + c)*(n/2))
    cilk_spawn mm_dac(X(C,0,0), n_C, X(A,0,0), n_A, X(B,0,0), n_B, n/2);
    cilk_spawn mm_dac(X(C,0,1), n_C, X(A,0,0), n_A, X(B,0,1), n_B, n/2);
    cilk_spawn mm_dac(X(C,1,0), n_C, X(A,1,0), n_A, X(B,0,0), n_B, n/2);
               mm_dac(X(C,1,1), n_C, X(A,1,0), n_A, X(B,0,1), n_B, n/2);
    cilk_sync;
    cilk_spawn mm_dac(X(C,0,0), n_C, X(A,0,1), n_A, X(B,1,0), n_B, n/2);
    cilk_spawn mm_dac(X(C,0,1), n_C, X(A,0,1), n_A, X(B,1,1), n_B, n/2);
    cilk_spawn mm_dac(X(C,1,0), n_C, X(A,1,1), n_A, X(B,1,0), n_B, n/2);
               mm_dac(X(C,1,1), n_C, X(A,1,1), n_A, X(B,1,1), n_B, n/2);
    cilk_sync;
  }
}
```

The base case is too small. We must coarsen the recursion to overcome function-call overheads.

Running time: 93.93s

⋯ about 50× **slower** than the last version!

Coarsening The Recursion

Just one tuning parameter, for the size of the base case.

```
void mm_dac(double *restrict C, int n_C,
            double *restrict A, int n_A,
            double *restrict B, int n_B,
            int n)
{ // C += A * B
  assert((n & (-n)) == n);
  if (n <= THRESHOLD) {
    mm_base(C, n_C, A, n_A, B, n_B, n);
  } else {
#define X(M,r,c) (M + (r*(n_ ## M) + c)*(n/2))
    cilk_spawn mm_dac(X(C,0,0), n_C, X(A,0,0), n_A, X(B,0,0), n_B, n/2);
    cilk_spawn mm_dac(X(C,0,1), n_C, X(A,0,0), n_A, X(B,0,1), n_B, n/2);
    cilk_spawn mm_dac(X(C,1,0), n_C, X(A,1,0), n_A, X(B,0,0), n_B, n/2);
               mm_dac(X(C,1,1), n_C, X(A,1,0), n_A, X(B,0,1), n_B, n/2);
    cilk_sync;
    cilk_spawn mm_dac(X(C,0,0), n_C, X(A,0,1), n_A, X(B,1,0), n_B, n/2);
    cilk_spawn mm_dac(X(C,0,1), n_C, X(A,0,1), n_A, X(B,1,1), n_B, n/2);
    cilk_spawn mm_dac(X(C,1,0), n_C, X(A,1,1), n_A, X(B,1,0), n_B, n/2);
               mm_dac(X(C,1,1), n_C, X(A,1,1), n_A, X(B,1,1), n_B, n/2);
    cilk_sync;
  }
}
```

Coarsening The Recursion

```
void mm_dac(double *restrict C, int n_C,
            double *restrict A, int n_A,
            double *restrict B, int n_B,
            int n)
{ // C += A * B
  assert((n & (-n)) == n);
  if (n <= THRESHOLD) {
    mm_base(C, n_C, A, n_A, B, n_B, n);
  } else {
#define X(M,r,c) (M + (r*(n_ ## M) + c)*(n/2))
    cilk_spawn mm_dac(X(C,0,0), n_C, X(A,0,0), n_A, X(B,0,0), n_B, n/2);
    cilk_spawn mm_dac(X(C,0,1), n_C, X(A,0,0), n_A, X(B,0,1), n_B, n/2);
    cilk_spawn mm_dac(X(C,1,0), n_C, X(A,1,0), n_A, X(B,0,0), n_B, n/2);
               dac(X(C,1,1), n_C, X(A,1,0), n_A, X(B,0,1), n_B, n/2);

               dac(X(C,0,0), n_C, X(A,0,1), n_A, X(B,1,0), n_B, n/2);
               dac(X(C,0,1), n_C, X(A,0,1), n_A, X(B,1,1), n_B, n/2);
               dac(X(C,1,0), n_C, X(A,1,1), n_A, X(B,1,0), n_B, n/2);
               dac(X(C,1,1), n_C, X(A,1,1), n_A, X(B,1,1), n_B, n/2);
```

```
void mm_base(double *restrict C, int n_C,
             double *restrict A, int n_A,
             double *restrict B, int n_B,
             int n)
{ // C = A * B
  for (int i = 0; i < n; ++i)
    for (int k = 0; k < n; ++k)
      for (int j = 0; j < n; ++j)
        C[i*n_C+j] += A[i*n_A+k] * B[k*n_B+j];
}
```

Coarsening The Recursion

Base-case size	Running time (s)
4	3.00
8	1.34
16	1.34
32	1.30
64	1.95
128	2.08

```
void mm_dac(double *restrict C, int n_C,
            double *restrict A, int n_A,
            double *restrict B, int n_B,
            int n)
{ // C += A * B
  assert((n & (-n)) == n);
  if (n <= THRESHOLD) {
    mm_base(C, n_C, A, n_A, B, n_B, n);
  } else {
#define X(M,r,c) (M + (r*(n_ ## M) + c)*(n/2))
    cilk_spawn mm_dac(X(C,0,0), n_C, X(A,0,0), n_A, X(B,0,0), n_B, n/2);
    cilk_spawn mm_dac(X(C,0,1), n_C, X(A,0,0), n_A, X(B,0,1), n_B, n/2);
    cilk_spawn mm_dac(X(C,1,0), n_C, X(A,1,0), n_A, X(B,0,0), n_B, n/2);
               mm_dac(X(C,1,1), n_C, X(A,1,0), n_A, X(B,0,1), n_B, n/2);
    cilk_sync;
    cilk_spawn mm_dac(X(C,0,0), n_C, X(A,0,1), n_A, X(B,1,0), n_B, n/2);
    cilk_spawn mm_dac(X(C,0,1), n_C, X(A,0,1), n_A, X(B,1,1), n_B, n/2);
    cilk_spawn mm_dac(X(C,1,0), n_C, X(A,1,1), n_A, X(B,1,0), n_B, n/2);
               mm_dac(X(C,1,1), n_C, X(A,1,1), n_A, X(B,1,1), n_B, n/2);
    cilk_sync;
  }
}
```

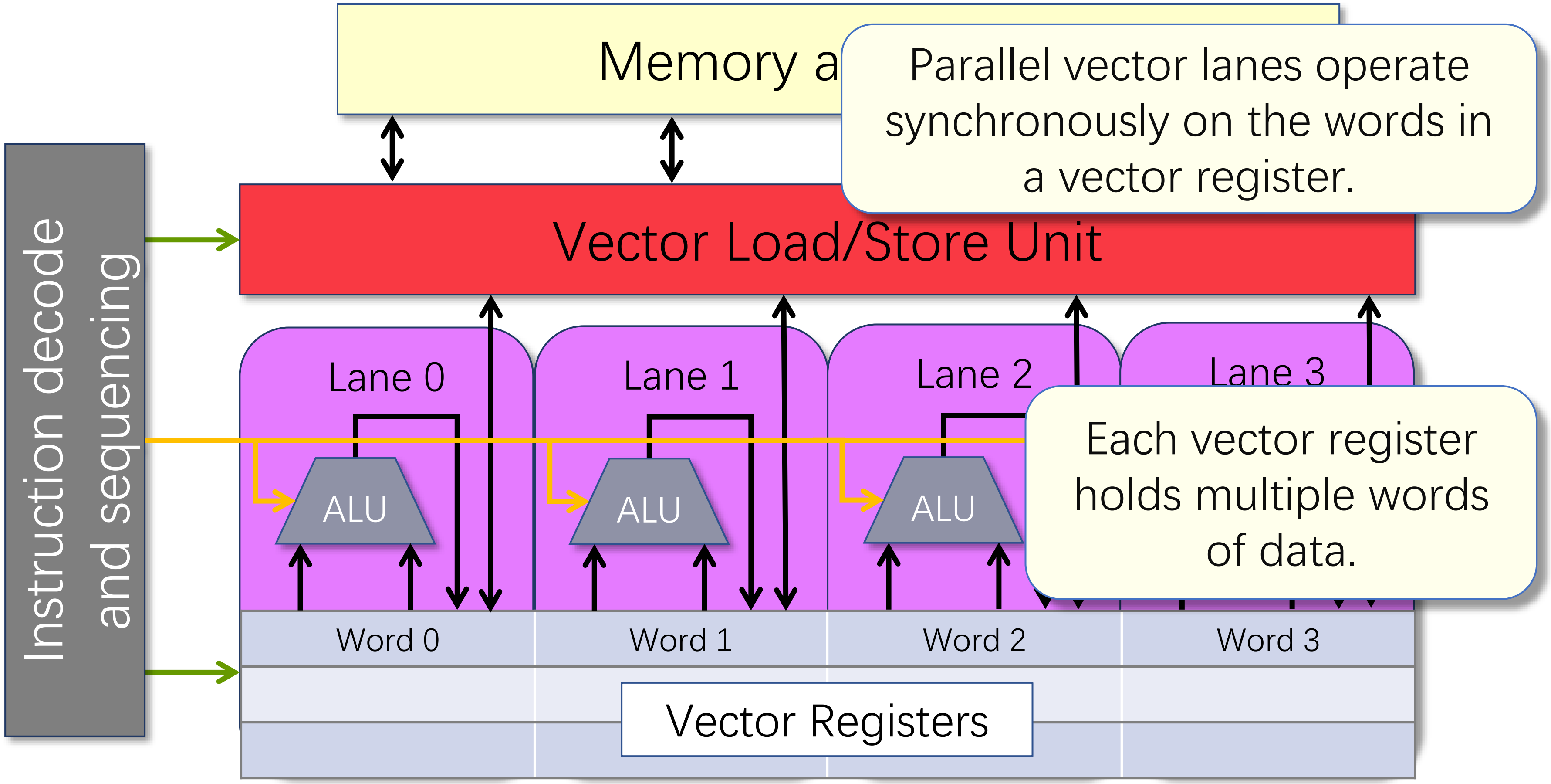
8. Divide-and-Conquer

Version	Implementation	Running time (s)	Relative speedup	Absolute Speedup	GFLOPS	Percent of peak
1	Python	21041.67	1.00	1	0.006	0.001
2	Java	2387.32	8.81	9	0.058	0.007
3	C	1155.77	2.07	18	0.118	0.014
4	+ interchange loops	177.68	6.50	118	0.774	0.093
5	+ optimization flags	54.63	3.25	385	2.516	0.301
6	Parallel loops	3.04	17.97	6,921	45.211	5.408
7	+ tiling	1.79	1.70	11,772	76.782	9.184
8	Parallel divide-and-conquer	1.30	1.38	16,197	105.722	12.646

Implementation	Cache references (millions)	L1-d cache misses (millions)	Last-level cache misses (millions)
Parallel loops	104,090	17,220	8,600
+ tiling	64,690	11,777	416
Parallel divide-and-conquer	58,230	9,407	64

Vector Hardware

Modern microprocessors incorporate vector hardware to process data in single-instruction stream, multiple-data stream (SIMD) fashion



Compiler Vectorization

Clang/LLVM uses vector instructions automatically when compiling at optimization level -O2 or higher

Can be checked in a *vectorization report* as follows:

```
$ clang -O3 -std=c99 mm.c -o mm -Rpass=vector
mm.c:42:7: remark: vectorized loop (vectorization width: 2,
interleaved count: 2) [-Rpass=loop-vectorize]
      for (int j = 0; j < n; ++j) {
      ^
```

Many machines don’t support the newest set of vector instructions, however, so the compiler uses vector instructions conservatively by default

Vectorization Flags

Programmers can direct the compiler to use modern vector instructions using **compiler flags** such as the following:

- **-mavx**: Use Intel AVX vector instructions
- **-mavx2**: Use Intel AVX2 vector instructions
- **-mfma**: Use fused multiply-add vector instructions
- **-march=<string>**: Use whatever instructions are available on the specified architecture
- **-march=native**: Use whatever instructions are available on the architecture of the machine doing compilation

Due to restrictions on floating-point arithmetic, additional flags, such as -ffast-math, might be needed for these vectorization flags to have an effect

Version 9: Compiler Vectorization

Version	Implementation	Running time (s)	Relative speedup	Absolute Speedup	GFLOPS	Percent of peak
1	Python	21041.67	1.00	1	0.006	0.001
2	Java	2387.32	8.81	9	0.058	0.007
3	C	1155.77	2.07	18	0.118	0.014
4	+ interchange loops	177.68	6.50	118	0.774	0.093
5	+ optimization flags	54.63	3.25	385	2.516	0.301
6	Parallel loops	3.04	17.97	6,921	45.211	5.408
7	+ tiling	1.79	1.70	11,772	76.782	9.184
8	Parallel divide-and-conquer	1.30	1.38	16,197	105.722	12.646
9	+ compiler vectorization	0.70	1.87	30,272	196.341	23.486

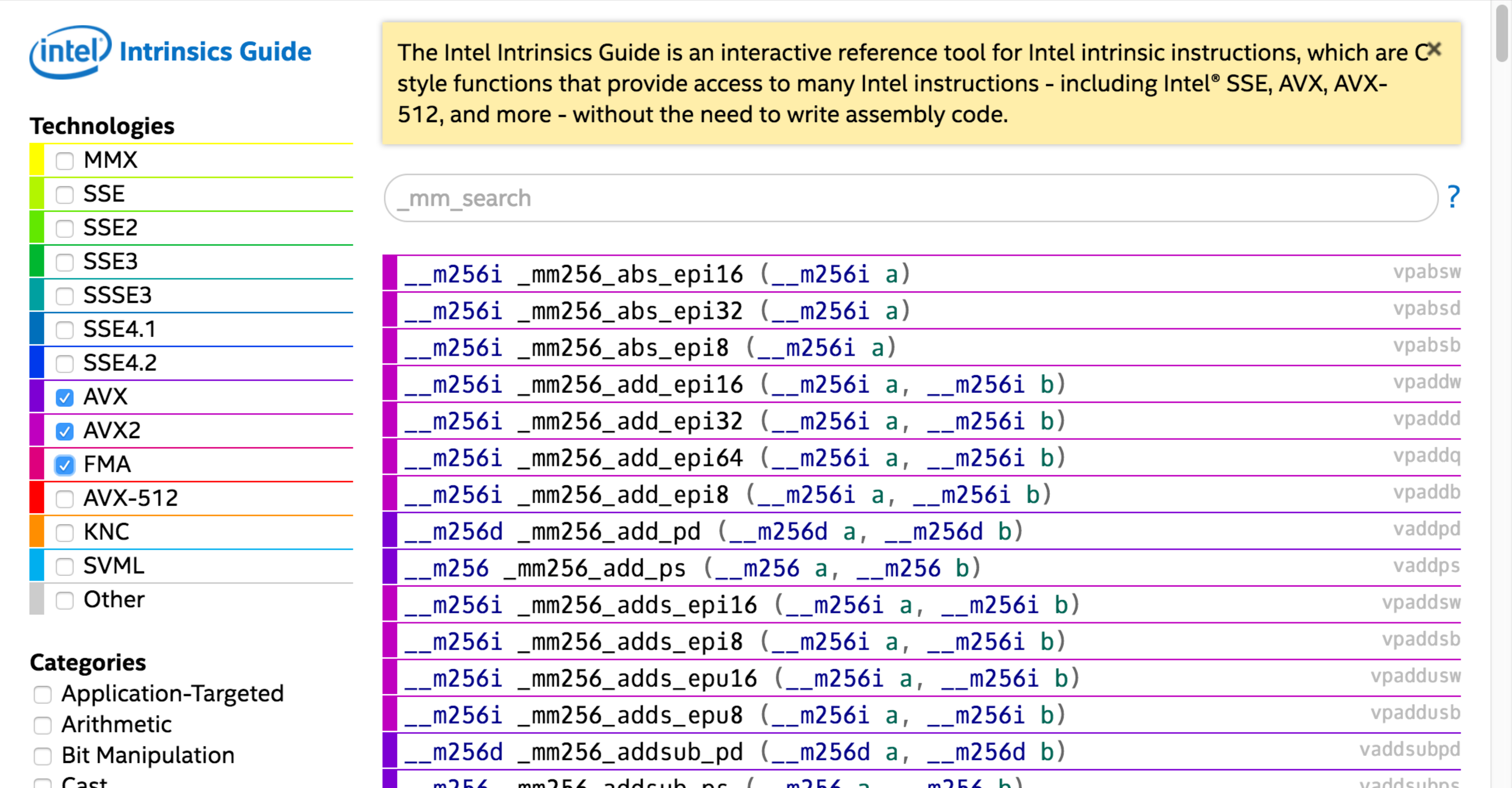
Using the flags `-march=native -ffast-math` nearly doubles the program's performance!

Can we be smarter than the compiler?

AVX Intrinsic Instructions

- **Intel provides C-style functions, called *intrinsic instructions*, that provide direct access to hardware vector operations:**

https://software.intel.com/sites/landingpage/IntrinsicsGuide/



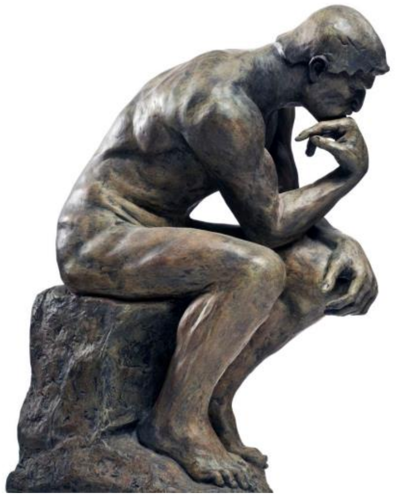
Plus More Optimizations

We can apply several more insights and performance-engineering tricks to make this code run faster, including:

- **Preprocessing**
- **Matrix transposition**
- **Data alignment**
- **Memory-management optimizations**
- **A clever algorithm for the base case that uses AVX intrinsic instructions explicitly**

Plus Performance Engineering

Think,



code,

run, run, run…

…to test and measure many different implementations

Version 10: AVX Intrinsics

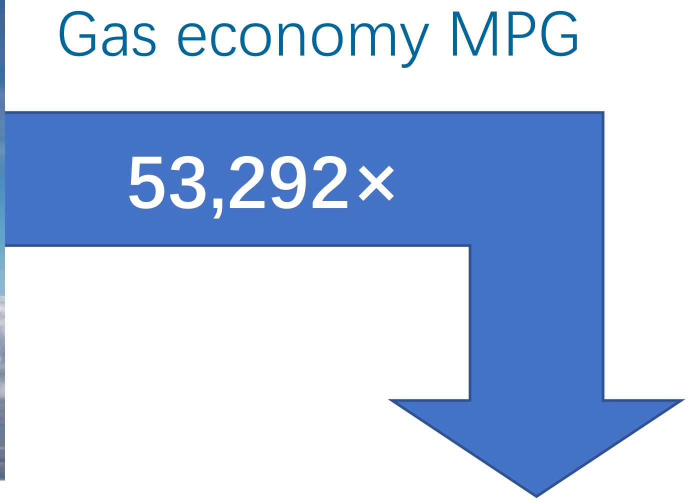
Version	Implementation	Running time (s)	Relative speedup	Absolute Speedup	GFLOPS	Percent of peak
1	Python	21041.67	1.00	1	0.006	0.001
2	Java	2387.32	8.81	9	0.058	0.007
3	C	1155.77	2.07	18	0.118	0.014
4	+ interchange loops	177.68	6.50	118	0.774	0.093
5	+ optimization flags	54.63	3.25	385	2.516	0.301
6	Parallel loops	3.04	17.97	6,921	45.211	5.408
7	+ tiling	1.79	1.70	11,772	76.782	9.184
8	Parallel divide-and-conquer	1.30	1.38	16,197	105.722	12.646
9	+ compiler vectorization	0.70	1.87	30,272	196.341	23.486
10	+ AVX intrinsics	0.39	1.76	53,292	352.408	41.677

Version 11: Final Reckoning

Version	Implementation	Running time (s)	Relative speedup	Absolute Speedup	GFLOPS	Percent of peak
1	Python	21041.67	1.00	1	0.006	0.001
2	Java	2387.32	8.81	9	0.058	0.007
3	C	1155.77	2.07	18	0.118	0.014
4	+ interchange loops	177.68	6.50	118	0.774	0.093
5	+ optimization flags	54.63	3.25	385	2.516	0.301
6	Parallel loops	3.04	17.97	6,921	45.211	5.408
7	+ tiling	1.79	1.70	11,772	76.782	9.184
8	Parallel divide-and-conquer	1.30	1.38	16,197	105.722	12.646
9	+ compiler vectorization	0.70	1.87	30,272	196.341	23.486
10	+ AVX intrinsics	0.39	1.76	53,292	352.408	41.677
11	Intel MKL	0.41	0.97	51,497	335.217	40.098

Version 10 is competitive with Intel's professionally engineered Math Kernel Library!

Engineering the Performance of your Algorithms



- **You won’t generally see the magnitude of performance improvement we obtained for matrix multiplication**
- **But in this course, you will learn how to print the currency of performance all by yourself**

Overall Structure in this Course

Performance Engineering	Algorithm Engineering
Parallelism	Sorting / Semisorting
I/O efficiency	Matrix multiplication
New Bentley rules	Graph algorithms
Brief overview of architecture	Geometry Algorithms

EE/CS217	GPU Architecture and Parallel Programming
CS211	High Performance Computing
CS213	Multiprocessor Architecture and Programming (Stanford CS149)
CS247	Principles of Distributed Computing