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Fractal image approximation and orthogonal bases

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Abstract

We are concerned with the fractal approximation of multidimensional functions in $\mathscr{L}^2$. In particular, we treat a position-dependent approximation using orthogonal bases of $\mathscr{L}^2$ and no search. We describe a framework that establishes a connection between the classic orthogonal approximation and the fractal approximation. The main theorem allows easy and univocal computation of the parameters of the approximating function. From the computational perspective, the result avoids to solve ill-conditioned linear systems that are usually needed in former fractal approximation techniques. Additionally, using orthogonal bases the most compact representation of the approximation is obtained. We discuss the approximation of gray-scale digital images as a direct application of our approximation scheme. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Some years ago it has been shown that deterministic fractal geometry is capable to produce very complex behaviors using apparently simple mathematical models [3]. In particular fractal models appeared suitable to represent real world images [6,14,20,21].

In 1987, Barnsley originally proposed to use deterministic fractal geometry to obtain a compressed representation of digital images. Some years later, one of his students devised the first algorithm capable to partially achieve that goal [10].

The idea of fractal coding is to represent the signal, or better, the function to be approximated, solely by the relations that are present between affinely transformed parts of the signal and the signal itself. Through the removal of 'self-affine redundancy', one hopes to obtain a more compact representation than the original one.

Barnsley [4], Jacquin [10–12] and Jacobs et al. [9] presented different methods for looking for the similarities present in digital images. For simplicity of implementation the search for similarities was performed only between *blocks* in which the image was initially decomposed. The brightness of a block

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was being approximated by a linear transformation of the brightness of another bigger block. Among all the bigger candidate blocks the one that best approximated the original was chosen, together with a particular transformation.

The whole image was hence represented through the relationship between blocks and by the coefficients of such brightness transformation. They originally chose linear transformations with a constant translation term with respect to the position inside the block. Although later many other strategies has been proposed (see e.g. [1]) the search process was always computationally very intensive.

Motivated by the desire to reduce substantially the computational cost, Monro and Dudbridge proposed a different approach in which the approximation is applied independently on each single block [16]. The basic method, although simple to implement and very fast, does not perform well. It constrains too strong auto-similarities inside the blocks that are generally not present in real-world images. To obtain a better quality of the approximation the authors propose to substitute the constant translation term with a polynomial in the pixel coordinates. The polynomial approximates the residual error that cannot be captured by the fractal approximation.

Barnsley himself introduced, in the one-dimensional case, a class of fractal interpolation functions which have a self-similarity property [2]. In this paper we want to show that it is possible to reformulate Barnsley's theory in terms of fractal approximation functions in $\mathscr{L}^2(R^n)$. In particular, since we are going to treat the problem of image coding (i.e., the approximation of a brightness function) we will consider, without loss of generality, the two-dimensional case.

In that framework we will describe a more general type of position-dependent approximation than the one by Monro and Dudbridge, in which the translation term is a function that belongs to the subspace generated by a particular orthogonal basis. Other techniques that use orthogonal basis, although developed from a different approach, can be found in [18,19].

The main result of this work is a theorem that builds the fractal approximation from an approximation of the gray-scale function expressed with respect to the same basis. Since the resulting approximation is optimal with respect to the chosen basis we will call it the best fractal orthogonal approximation (BFOA).

In practice, if we suppose to have a 'classic' place-dependent approximation the rules of the theorem 'turn it' into a fractal approximation. In this way, we avoid using heavy numerical methods to overcome the ill-conditioned problems associated to the type of polynomials used in [15–17].

We will show some results on the approximation of digital images obtained with cosine and Haar basis. We want to emphasize that the initial approximation can be computed with any algorithm, for example with fast technique like FFT or DWT. However, this work proposes a new approximation model, and not yet a compression technique.

Section 2 recalls some notations used in the rest of the paper, while Section 3 introduces a theory for fractal approximation in $\mathscr{L}^2(R^2)$ with a variant of the Collage theorem. Section 4 presents the main result of the BFOA and an issue on the contractivity of the operator.

The application of BFOA to image approximation is described in Section 5, where we show the results of using different orthogonal bases in a block coding framework. Once we fix the number of parameters, our approach gives a lower reconstruction error than the original Monro and Dudbridge polynomial approximation. In the same section we analyze the best splitting point heuristics as a searching method and we show the advantages given by the utilization of bigger blocks than the ones generally used.

2. Notations

We briefly recall some notation used in the paper. We consider functions in $\mathscr{L}^p$ with the metric

$$d(f,g) = \|f-g\|_p,$$

where

$$\|f\|_p = \left(\int_{R^2} |f(x)|^p \,\mathrm{d}\mu\right)^{1/p}.$$

Let f be a function in $\mathscr{L}^p$ and U a subspace of $\mathscr{L}^p$. With *best approximation* of f in U we define the function $f^* \in \mathscr{L}^p$ that satisfies

$$\|f - f^*\|_p = \inf_{g \in U} \|f - g\|_p.$$

In other words, f is the function that achieves the minimum distance from f^* with respect to the particular norm chosen.

We recall that if U is a finite-dimensional subspace, then there exists at least one best approximation. In particular, if U is generated by an orthogonal basis $\{u_0, u_1, \dots, u_n\}$, then the best approximation of f is given by $\sum_{j=0}^{n} c_j u_j$ and it belongs to U if and only if $c_j = (\|u_j\|_2^2)^{-1}\langle f, u_j\rangle$. The expression is called Fourier series and c_j are the Fourier coefficients.

In the rest of the paper we will assume $p = 2$ because of the important properties of $\mathscr{L}^2$ and because the $\mathscr{L}^2$-norm is the easiest norm to manipulate. We will denote by $\langle \cdot, \cdot \rangle$ the scalar product in $\mathscr{L}^2$ defined as $\langle f, f \rangle = \|f\|_2^2$, with $\circ$ the operator that composes two functions and with $T^n(f)$ the iterated application of T to f, n times.

3. Fractal approximation in $\mathscr{L}^2(R^2)$

We identify a continuous gray-scale image with a function $f \in \mathscr{L}^2$ whose domain is a compact set A, attractor of an IFS $\{A;\, w_1, \dots, w_N\}$ (see [3])

$$A = \bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} w_i(A),$$

where the maps w_i are affine, contractive and non-overlapping,[1] i.e., $w_i(x) = L_i x + \tau_i$, $x \in R^2$, and where L_i are 2×2 scaling matrices, and τ_i are translation vectors. The maps w_i describe the underlying 'geometry' of the domain A of the function f.

A fractal approximation of f is a function f^* associated with an $\mathscr{L}^2$-contractive operator such that f^* is the unique fixed point of T, that is $Tf^* = f^*$.

Since the metric space $(\mathscr{L}^2(A), \|\cdot\|_2)$ is complete, by the Banach's theorem there is only one fixed point that can be obtained by the following *reconstruction algorithm*

$$\lim_{n \to \infty} T^n(g) = f^*, \quad \forall g \in \mathscr{L}^2(A). \tag{1}$$

The procedure (1) permits to obtain f^* by the iterations of the operator T starting from *any* initial function g.

The operator T is usually built from the IFS maps w_i and from some appropriate functions F_i:

$$(Th)(x) = F_i(h(w_i^{-1}(x))), \quad \forall x \in A_i, \quad h \in \mathscr{L}^2(A).$$

In a more general case, we can consider a *place-dependent* operator

$$(Th)(x) = F_i(w_i^{-1}(x), h(w_i^{-1}(x))), \quad \forall x \in A_i,\ h \in \mathscr{L}^2(A), \tag{2}$$

where $F_i : A \times [c, d] \to [c, d]$, $i = 1, \dots, N$, are functions satisfying the Lipschitz condition

$$|F_i(x, y_1) - F_i(x, y_2)| \leqslant s_i |y_1 - y_2|, \quad s_i > 0,\ \forall x \in A,\ \forall y_1, y_2 \in [c, d].$$

When $(\sum_{i=1}^{N} |\det L_i|\, s_i^2)^{1/2} < 1$ then T is contractive in $\mathscr{L}^2(A)$. Indeed, since the sets A_i are disjoint, we have

$$\begin{aligned}\|Th - Tg\|_2^2 &= \sum_{i=1}^{N} \int_{A_i} |F_i(w_i^{-1}(x), h(w_i^{-1}(x))) - F_i(w_i^{-1}(x), g(w_i^{-1}(x)))|^2 \, d\mu \\ &= \sum_{i=1}^{N} |\det L_i| \int_A |F_i(x, h(x)) - F_i(x, g(x))|^2 \, d\mu \\ &\leqslant \sum_{i=1}^{N} |\det L_i| s_i^2 \int_A |h(x) - g(x)|^2 \, d\mu \\ &= \sum_{i=1}^{N} |\det L_i| s_i^2 \, \|h - g\|_2^2.\end{aligned} \tag{3}$$

Of particular interest is the linear case

$$F_i(x, y) = \alpha_i y + q_i(x), \quad \alpha_i \in R, \quad q_i \in \mathscr{L}^2(A), \tag{4}$$

in which the operator T becomes

$$(Th)(x) = \alpha_i h(w_i^{-1}(x)) + q_i(w_i^{-1}(x)), \quad \forall x \in A_i,\ h \in \mathscr{L}^2(A). \tag{5}$$

From Eq. (3) it follows that if $(\sum_{i=1}^{N} |\det L_i| \alpha_i^2)^{1/2} < 1$ then T is contractive in $\mathscr{L}^2(A)$. It is interesting to remark that T can be

[1] $w_j(A) \cap w_k(A) = \emptyset \quad \forall j, k = 1, \dots, N$ with $j \neq k$.

contractive even if not all of the maps satisfy $|\alpha_i| < 1$.

3.1. The inverse problem

The problem of finding a contractive operator T whose fixed point is f, or better, close to f, is called the *inverse problem*. The Collage theorem provides directions on how to evaluate a given operator.

Theorem 1. *Let $f \in \mathscr{L}^2(A)$, $T : \mathscr{L}^2(A) \to \mathscr{L}^2(A)$ be the contractive operator defined in* Eq. (2) *with contractivity factor $0 \leqslant K < 1$, and f^* its fixed point. If*

$$\|f - Tf\|_2 < \varepsilon,$$

then

$$\|f^* - f\|_2 < \frac{\varepsilon}{1-K},$$

or equivalently,

$$\|f^* - f\|_2 < (1-K)^{-1}\|f - Tf\|_2.$$

Proof. The proof proceeds as for the classic IFS theory [3]. □

The theorem states that once we are given with a particular $f \in \mathscr{L}^2(A)$ to approximate, if T is such that f and its image under T are 'near enough', then f will be 'near enough' to f^*.

Note that the theorem is not constructive. It provides a measure of the quality of the approximation without having to compute the fixed point of T, but it does not suggest any method to find an explicit form of T.

3.2. Best fractal approximation in $\mathscr{L}^2$

In the rest of the paper we consider the case of the linear operator described in Eq. (5). In particular, we assume that the functions q_i have been chosen in a subspace U of $\mathscr{L}^2(A)$.

We call *best fractal approximation* of f the fixed point f^* of the operator T such that f has minimum distance from Tf. That is,

$$\inf_{q_i \in U,\, K<1} \|f - (\alpha_i f \circ w_i^{-1} + q_i \circ w_i^{-1})\|_2,$$

where $K = (\sum_{i=1}^{N} |\det L_i| \alpha_i^2)^{1/2}$.

Alternatively, the search for the best fractal approximation of f can be carried out by looking for the parameters α_i and the functions q_i which minimize

$$\|f \circ w_i - (\alpha_i f + q_i)\|_2^2 \quad \forall i = 1, \dots, N, \tag{6}$$

since

$$\begin{aligned}\|f - (\alpha_i f \circ w_i^{-1} + q_i \circ w_i^{-1})\|_2^2 &= \sum_{i=1}^{N} \int_{A_i} |f(x) - (\alpha_i f(w_i^{-1}(x)) + q_i(w_i^{-1}(x)))|^2 \,\mathrm{d}\mu \\ &= \sum_{i=1}^{N} |\det L_i| \int_{A} |f(w_i(x)) - (\alpha_i f(x) + q_i(x))|^2 \,\mathrm{d}\mu \\ &= \sum_{i=1}^{N} |\det L_i| \, \|f \circ w_i - (\alpha_i f + q_i)\|_2^2.\end{aligned}$$

4. The best fractal orthogonal approximation

When the functions q_i belong to a subspace U generated by an orthogonal basis $\{u_0, u_1, \dots, u_n\}$, the operator T can be obtained by fairly simple rules.

The following theorem allows us to construct the function that approximates $f \circ w_i$, i.e., the function that minimizes $\|f \circ w_i - (\alpha_i f + q_i)\|_2$ with respect to $q_i \in U$, $\alpha_i \in R$. We call such approximating function the *best fractal orthogonal approximation* of $f \circ w_i$ in U.

Theorem 2. *Let $f \in \mathscr{L}^2(A)$, $A \subset R^2$ be a compact set with $\mu(A) < +\infty$ and $\{w_1, \dots, w_N\}$ be non-overlapping contractive affine maps, such that A is the attractor of the associated IFS. Let $\{u_0, u_1, \dots, u_n\}$ denote an orthogonal system in $\mathscr{L}^2(A)$, U the subspace generated by its elements, $\sum_{j=0}^{n} c_j u_j$ and $\sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j$ the best approximation in U of f and $f \circ w_i$, respectively. Then, for each $i = 1, \dots, N$, there is an element in U,*

$g_i = \sum_{j=0}^{n} \varphi_j^{(i)} u_j$, *univocally defined by*

$$\varphi_j^{(i)} = \tilde{c}_j^{(i)} - \hat{\alpha}_i c_j, \quad \hat{\alpha}_i = \frac{\int_A f \cdot f \circ w_i \, \mathrm{d}\mu - \sum_{j=0}^{n} \|u_j\|_2^2 c_j \tilde{c}_j^{(i)}}{\int_A f^2 \, \mathrm{d}\mu - \sum_{j=0}^{n} \|u_j\|_2^2 c_j^2} \tag{7}$$

such that $\hat{\alpha}_i f + g_i$, $i = 1, \dots, N$, *is the best fractal orthogonal approximation of* $f \circ w_i$ *in* U. *In other words, for any* i, $\hat{\alpha}_i f + g_i$ *has the minimum distance from* $f \circ w_i$ *with respect to chosen orthogonal basis.*

If $\int_A f^2 \, \mathrm{d}\mu - \sum_{j=0}^{n} \|u_j\|_2^2 c_j^2 = 0$, *we have to set* $\hat{\alpha}_i = 0$; *in this case the functions* g_i *correspond with the best approximation of* $f \circ w_i$ *in* U.

Proof. First of all, note that if $\sum_{j=0}^{n} c_j u_j$ and $\sum_{j=0}^{n} d_j u_j$ are, respectively, the best approximation of f and g, with respect to the orthogonal basis $\{u_j\}_{j=0,\dots,n}$, we have

$$\left\langle f - \sum_{j=0}^{n} c_j u_j, g - \sum_{j=0}^{n} d_j u_j \right\rangle = \langle f, g \rangle - \sum_{j=0}^{n} \|u_j\|_2^2 c_j d_j. \tag{8}$$

The reason is explained in the following derivation:

$$\left\langle f - \sum_{j=0}^{n} c_j u_j, g - \sum_{j=0}^{n} d_j u_j \right\rangle = \langle f, g \rangle - \sum_{j=0}^{n} d_j \langle f, u_j \rangle - \sum_{j=0}^{n} c_j \langle g, u_j \rangle + \sum_{j=0}^{n} \sum_{k=0}^{n} c_j d_k \langle u_j, u_k \rangle = \langle f, g \rangle - \sum_{j=0}^{n} \|u_j\|_2^2 c_j d_j,$$

where

$\langle u_j, u_k \rangle = 0$, then $k \neq j$, $\quad \langle u_j, u_j \rangle = \|u_j\|_2^2$,

$c_j = (\|u_j\|_2^2)^{-1} \langle f, u_j \rangle, \quad d_j = (\|u_j\|_2^2)^{-1} \langle g, u_j \rangle$,

$j = 0, \dots, n, \quad \|u_0\|_2^2 = \int_A \mathrm{d}\mu = \mu(A)$,

and where u_0 is the unitary function.

Having fixed the coefficients α_i, let $g_i = \sum_{j=0}^{n} \varphi_j^{(i)} u_j$ be the best approximation in U of $f \circ w_i - \alpha_i f$ (see Section 2). Therefore, $\varphi_j^{(i)}$ are the Fourier coefficients of $f \circ w_i - \alpha_i f$,

$$\varphi_j^{(i)} = (\|u_j\|_2^2)^{-1} \langle f \circ w_i - \alpha_i f, u_j \rangle.$$

Define now an auxiliary function

$$G(\alpha_i) = \| f \circ w_i - (\alpha_i f + g_i) \|_2^2 = \| (f \circ w_i - \alpha_i f) - g_i \|_2^2. \tag{9}$$

It is well known that

$$G(\alpha_i) = \min_{q_i \in U} \| (f \circ w_i - \alpha_i f) - q_i \|_2^2.$$

In addition to this, the minimum is unique. It follows that

$$\varphi_j^{(i)} = (\|u_j\|_2^2)^{-1} \langle f \circ w_i, u_j \rangle - \alpha_i (\|u_j\|_2^2)^{-1} \langle f, u_j \rangle = \tilde{c}_j^{(i)} - \alpha_i c_j,$$

and hence

$$g_i = \sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j - \alpha_i \sum_{j=0}^{n} c_j u_j,$$

where

$$c_j = (\|u_j\|_2^2)^{-1} \langle f, u_j \rangle, \quad \tilde{c}_j^{(i)} = (\|u_j\|_2^2)^{-1} \langle f \circ w_i, u_j \rangle.$$

Substituting in Eq. (9) we have

$$G(\alpha_i) = \left\| (f \circ w_i - \alpha_i f) - \left(\sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j - \alpha_i \sum_{j=0}^{n} c_j u_j \right) \right\|_2^2 = \left\| \left(f \circ w_i - \sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j \right) - \alpha_i \left(f - \sum_{j=0}^{n} c_j u_j \right) \right\|_2^2 = \int_A \left[\left(f \circ w_i - \sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j \right) - \alpha_i \left(f - \sum_{j=0}^{n} c_j u_j \right) \right]^2 \mathrm{d}\mu.$$

Now, once we consider G as a function of α_i, we can see that it is differentiable and it has a unique stationary point given by the solution of

$$\int_A \left(f - \sum_{j=0}^{n} c_j u_j \right) \left(f \circ w_i - \sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j \right) \mathrm{d}\mu = \alpha_i \int_A \left(f - \sum_{j=0}^{n} c_j u_j \right)^2 \mathrm{d}\mu. \tag{10}$$

By Eq. (8) we have $G^{(2)}(\alpha_i) = 2(\int_A f^2 \, \mathrm{d}\mu - \sum_{j=0}^{n} \|u_j\|_2^2 c_j^2)$. Using the Bessel's inequality and the hypothesis of the theorem it follows that $G^{(2)}(\alpha_i) > 0$, which means the point is a minimum. Eq. (10) can be rewritten as follows:

$$\alpha_i\left\langle f - \sum_{j=0}^{n} c_j u_j, f - \sum_{j=0}^{n} c_j u_j \right\rangle$$

$$= \left\langle f - \sum_{j=0}^{n} c_j u_j, f \circ w_i - \sum_{j=0}^{n} \tilde{c}_j^{(i)} u_j \right\rangle.$$

Finally, using Eq. (8), we have

$$\alpha_i\left(\langle f, f\rangle - \sum_{j=0}^{n} \|u_j\|_2^2 c_j^2\right)$$

$$= \langle f, f\circ w_i\rangle - \sum_{j=0}^{n} \|u_j\|_2^2 c_j \tilde{c}_j^{(i)},$$

and we obtain, as a solution, the value $\hat{\alpha}_i$ in Eq. (7). □

4.1. Remark

Actually, we do not have any guarantee that by using Eq. (7) we will satisfy the condition $K < 1$ since the parameters are obtained through an unconstrained minimization. However, we have always verified experimentally the contractivity condition in our test cases with different approximation orders, i.e., choices of n. The general results resisted all our attempts to prove it formally. However, under some particular condition, we can show that the operator T obtained by Eq. (7) is contractive in $\mathscr{L}^2(A)$.

Proposition 3. *Let $f \in \mathscr{L}^2(A)$ be a non-negative function with $\|f\|_2 \neq 0$. When considering the zero-order approximation, i.e. $n = 0$, and $\alpha_i \geqslant 0$, $\varphi_0^{(i)} \geqslant 0$, $\varphi_0^{(i)}$ are not all zero, the operator T defined by the parameters of* Eq. (7) *is a contraction in $\mathscr{L}^2(A)$.*

Proof. Rewriting Eq. (7) for $j = 0$ we have

$$\alpha_i(\|f\|_2^2 - c_0^2) = \langle f, f\circ w_i\rangle - c_0\,\tilde{c}_0^{(i)},$$

and hence

$$\alpha_i\,\|f\|_2^2 + \|f\|_1 \varphi_0^{(i)} = \langle f, f\circ w_i\rangle,$$

where $\varphi_0^{(i)} = \tilde{c}_0^{(i)} - \alpha_i c_0$, and f is non-negative. By the Schwartz's inequality,

$$\alpha_i \|f\|_2^2 + \|f\|_1 \varphi_0^{(i)} \leqslant \|f\|_2 \cdot \|f\circ w_i\|_2. \tag{11}$$

For convenience of notation, we indicate $D_i = |\det L_i|$. Squaring both sides in Eq. (11), multiplying by D_i and summing over i we get

$$\|f\|_2^4 \sum_{i=1}^{N} D_i\alpha_i^2 + \|f\|_1^2 \sum_{i=1}^{N} D_i(\varphi_0^{(i)})^2$$

$$+ 2\,\|f\|_2^2\|f\|_1 \sum_{i=1}^{N} D_i\alpha_i\,\varphi_0^{(i)} \leqslant \|f\|_2^4,$$

since

$$\|f\|_2^2 = \sum_{i=1}^{N} D_i\,\|f\circ w_i\|_2^2.$$

Hence,

$$\|f\|_2^4\left(\sum_{i=1}^{N} D_i\alpha_i^2 - 1\right) + \|f\|_1^2 \sum_{i=1}^{N} D_i(\varphi_0^{(i)})^2$$

$$+ 2\,\|f\|_2^2\|f\|_1 \sum_{i=1}^{N} D_i\alpha_i\varphi_0^{(i)} \leqslant 0.$$

When $\alpha_i \geqslant 0$, $\varphi_0^{(i)} \geqslant 0$ and $\varphi_0^{(i)}$ are all not zero, the left member is strictly greater than $\|f\|_2^4(\sum_{i=1}^{N} D_i\alpha_i^2 - 1)$.

Finally, we have

$$\|f\|_2^4\left(\sum_{i=1}^{N} D_i\alpha_i^2 - 1\right) < 0,$$

that guarantees the required contractivity since $\|f\|_2 \neq 0$ by hypothesis. □

5. Applications to block image coding

In order to apply the BFOA to image coding we consider the image decomposed in square blocks of 8×8 or 16×16 pixels. A single block becomes the domain A of the brightness function f, the function we want to approximate. We choose w_1, w_2, w_3, w_4 as the functions that map a square in its four equal sub-quadrants. Later in this section we will discuss a more general subdivision of blocks.

We first choose Legendre and Chebychev polynomials as orthogonal system $\{u_i\}_{i=0,\dots,n}$. However, the best results were obtained with the cosine basis which is briefly described in appendix A. Also we did some experiments using the Haar wavelet basis (see Appendix A).

For each block we compute from Eq. (7) the coefficients α_i and $\varphi_j^{(i)}$, $i = 1, \dots, 4$, which build the best fractal orthogonal approximation. The encoding of the block is represented only by these coefficients. The problem of quantization of the encoding is outside the scope of this paper. Our work proposes a new approximation model, and not yet a compression technique.

However, we compare our results with a similar place-dependent method which uses standard polynomials for the q_i, called the Bath Fractal Transform [15–17], when no search is performed. In the BFT, the authors obtain the coefficients of the polynomials and the value of a coefficient that plays the role of α_i by a least-squares optimization performed by a numerical resolution of linear systems.

In Tables 1–3 and in Fig. 1 we show the approximation error evaluated in the $\mathscr{L}^2$-norm for 'Lena' (Fig. 3) using the BFOA with different orthogonal bases. When the number of parameters are equal

Table 1
RMS error for 'Lena' using Monro–Dudbridge polynomial. Blocks are 8×8 pixels

Order	Parameters	RMS
0	2	7.73009
1	4	4.81795
2	6	4.01304
3	8	3.80255

Table 2
RMS error for 'Lena' using cosine basis. Blocks are 8×8 pixels. By $*$ we indicate a choice of intermediate order. The order is m as in Eq. (12)

Order	Parameters	RMS
0	2	7.73009
1	4	4.75527
*	6	3.42942
2	7	3.09499
*	8	2.73753
3	11	2.05501

Table 3
RMS error for 'Lena' using Haar basis. Blocks are 8×8 pixels. The resolution factor is m as in Eq. (13)

Resolution factor	Parameters	RMS
0	2	7.73009
1	3	7.10770
2	5	5.00614
3	9	4.17559
4	17	0.00000

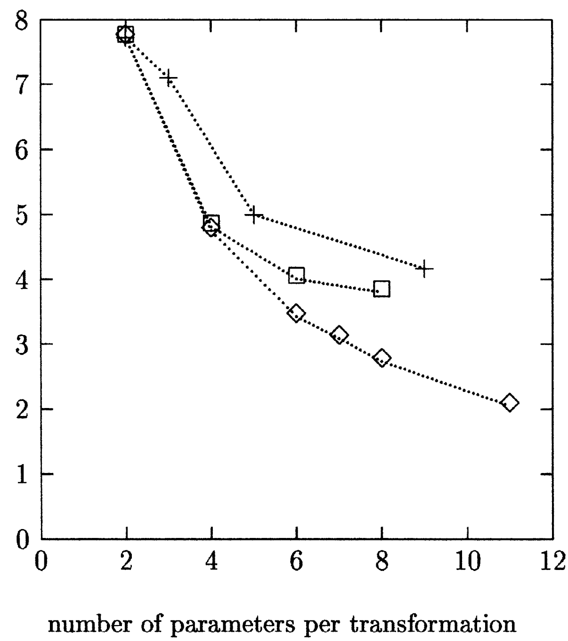

Fig. 1. RMS error for 'Lena' choosing different bases: ◇ cosine basis, + Haar basis, □ Monro–Dudbridge polynomial.

our approach gives a reconstruction error lower than the polynomial approximation in [15]. Additionally, not having ill-conditioned problems that are instead present in the BFT, it is possible to get an approximation error as low as we want by simply raising the order.

Finally, in order to evaluate the benefits of a more general tiling of the blocks, we implemented an adaptive searching of the best splitting point of the IFS that describes the domain A. The underlying motivation is to understand how much a better representation, in the fractal sense, of the domain of f could contribute to a better overall approximation.

Let us assume that the block dimension is $P \times P$ pixels. Each choice of (r, s) in the set of admissible points

$$\mathscr{D} = \left\{ \left(\frac{k_1}{P}, \frac{k_2}{P} \right) \middle| (k_1, k_2) \in [1, \ldots, P-1] \times [1, \ldots, P-1] \right\},$$

defines an IFS $\{A, w_1, w_2, w_3, w_4\}$ with attractor A. An example of a particular choice $(r, s) \in \mathscr{D}$ and the corresponding maps w_i is shown in Fig. 2. We consider the best splitting point the one whose IFS minimizes Eq. (6).

The optimization problem in $\mathscr{D}$ is solved through a gradient descent method, starting from the center of the block. We verified that our algorithm converges always to the global minimum and that checks on average one-fourth of the cardinality of $\mathscr{D}$.

Table 4 summarizes our experiments that employ the above strategy. Since we can afford to increase the order of the approximation we can safely have a bigger block dimension. If one chooses, for example, 16×16 blocks with 16 parameters per transformation, we have the same ratio data/parameters as for the 8×8 blocks with 4 parameters per transformation, but the reconstruction error is lower (cf. Figs. 4–6).

6. Conclusions

We introduced a general theory for the position-dependent fractal approximation of functions in

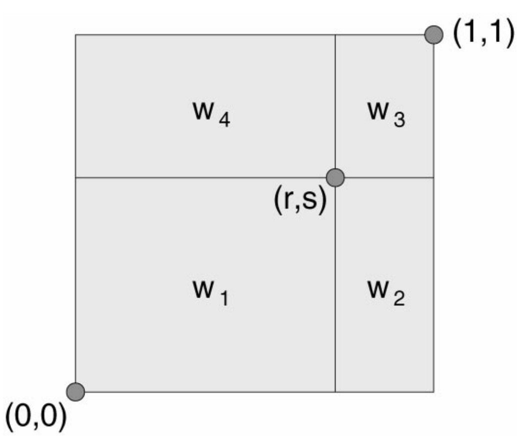

Fig. 2. The subdivision of A in $w_i(A)$.

Fig. 3. 'Lena' digitized 512×512, 8 bit per pixel.

Table 4
RMS error for 'Lena' with searching for the best (r, s)

Block size	Basis	Parameters	Data/total parameters	RMS
8×8	Haar	4	4 : 1	5.36103
8×8	Cosine	4	4 : 1	4.34129
16×16	Cosine	16	4 : 1	3.80247

Fig. 4. Blocks 8×8 pixels, cosine basis, 4 parameters per transformation with searching for the best (r, s), RMS $= 4.34129$.

Fig. 5. Blocks 8×8 pixels, Haar basis, 4 parameters per transformation with searching for the best (r, s), RMS $= 5.36103$.

$\mathscr{L}^2(R^n)$, called the 'best fractal orthogonal approximation', that connects IFS and orthogonal bases. Loosely speaking, our method is capable of 'trans-

Fig. 6. Blocks 16×16 pixels, cosine basis, 16 parameters per transformation with searching for the best (r, s), RMS $= 3.80247$.

forming' a classic place-dependent approximation into a fractal approximation.

Our approach can be very useful in multidimensional signal processing. In particular, we showed an application to two-dimensional discrete data, and specifically to digital images. Compared with other position-dependent approximation described in [15–17] it yields better quality of the approximation and less computational efforts.

More adaptive geometry, better methods of searching block similarities and more adaptive functional approximation seems to be the main goals of the future progresses in fractal image compression.

Recent papers [5,7,13,22] propose to search the similarity relations in the wavelet domain of the images. The results are comparable to state-of-the-art methods for image coding and they are attracting new research interests.

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Appendix A. Some remarks on the orthogonal bases used

In the following we outline the two prominent basis we used in our experimentations: the cosine basis and the Haar wavelet basis.

A.1. Cosine basis

It is well known that the functions

$$u_i(x)\,u_j(y), \quad i,j = 0,1,\ldots, \text{ where } u_k(t) = \cos k\pi t,$$

form a complete orthogonal system on the set $I^2 = [0,1] \times [0,1]$.

The decomposition of $f(x, y)$ of order m is given by

$$\sum_{i=0}^{m} \sum_{j=0}^{m-i} a_{ij} u_i(x)\,u_j(y) = \sum_{0 \leqslant i+j \leqslant m} a_{ij} u_i(x)\,u_j(y), \qquad (12)$$

where

$$a_{ij} = \frac{1}{h_{ij}} \int_0^1 \int_0^1 f(x, y) u_i(x)\,u_j(y)\,\mathrm{d}x\,\mathrm{d}y,$$

with

$$h_{ij} = \int_0^1 \int_0^1 u_i^2(x) u_j^2(y)\,\mathrm{d}x\,\mathrm{d}y = \begin{cases} 1 & i = 0 = j, \\ 1/4 & i, j > 0, \\ 1/2 & i = 0, \text{ or } j = 0,\ i \neq j. \end{cases}$$

A.2. Haar basis

Theorem 2 is also remarkable because it allows to use wavelets. We choose the following orthogonal basis of elementary wavelets – a set of function generated by dilation and translation of single function ψ, called the 'mother wavelet' –

$$\psi_{jk}(x) = 2^{j/2}\,\psi(B^j x - k), \quad x \in R^2,\ j \in Z,\ k \in Z^2,$$

where as mother wavelet we choose

$$\psi(x) = \begin{cases} 1 & x \in [0,1] \times [0,1/2), \\ -1 & x \in [0,1] \times [1/2,1], \end{cases}$$

and the matrix B, called matrix dilation, is

$$\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}.$$

This two-dimensional orthogonal basis of wavelets can be considered a generalization in $\mathscr{L}^2(R^2)$ of the Haar's system [8].

If one chooses the orthogonal basis ψ_{jk}, the decomposition of $f(x)$ at the resolution factor m is given by

$$\sum_{j=0}^{m} \sum_{k \in Z^2} a_{jk} \psi_{jk}(x), \quad x \in I^2, \qquad (13)$$

where the coefficients a_{jk} are

$$a_{jk} = \int_{I^2} f(x) \psi_{jk}(x)\,\mathrm{d}x.$$

The sum in Eq. (13) is taken over multi-index k such that $x \in I^2$.

Note that the integrals involved in the computation of the coefficients, with respect to the bases described above, can be easily implemented as discrete summation. Alternatively, it is possible to compute directly the coefficients by using efficient algorithms like the FFT for the cosine basis, or the DWT for wavelets.

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