1 Euler's Method and Midpoint Method - Error

$\dot{x}(t) = f(x(t), t)$

## Euler's Method

$$x(t+\Delta t) \cong x(t) + \Delta t\, f(x,t)$$

Taylor Series expansion of $x(t+\Delta t)$ about $t$:

$$x(t+\Delta t) = \underbrace{\underbrace{x(t)}_{\text{constant term}} + \underbrace{\Delta t\, \dot{x}(t)}_{\text{linear term}}}_{\text{Euler's Method is equivalent to approximating } x(t+\Delta t) \text{ with the constant + linear terms}} + \underbrace{\underbrace{O(\Delta t^2)}_{\text{higher order terms}}}_{\text{Euler's method drops these higher order terms, hence making an error of magnitude } O(\Delta t^2) \text{ in each time step}}$$

The error in one Euler step is $O(\Delta t^2)$
This is called the local truncation error

To reach some time $T$ in the future, we must take $\frac{T}{\Delta t}$ number of steps

Therefore the accumulated error is

$$\frac{T}{\Delta t} \cdot O(\Delta t^2) = O(\Delta t)$$

Thus Euler's Method is said to be first order accurate.

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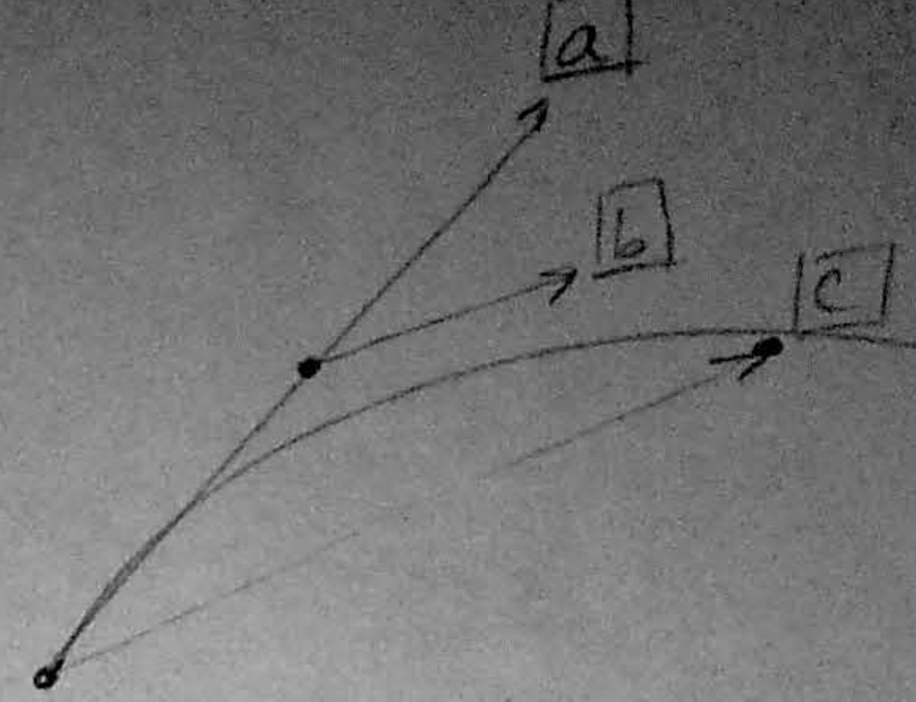
3 Any method which is first order accurate or more accurate is said to be consistent with the differential equation.

Euler's Method and the Midpoint Method are both consistent.
The Midpoint Method is more accurate.

## Midpoint Method




$\boxed{a}$ $\quad \Delta x = \Delta t\, f(x,t)$

$\boxed{b}$ $\quad f_{mid} = f\left(x + \frac{\Delta x}{2},\ t + \frac{\Delta t}{2}\right)$

$\boxed{c}$ $\quad x(t+\Delta t) \cong x(t) + \Delta t\, f_{mid}$

### Taylor Series expansion of $f(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2})$ about $(x,t)$

$$f\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right) = f\left(x, t+\frac{\Delta t}{2}\right) + \frac{\Delta x}{2} f_x\left(x, t+\frac{\Delta t}{2}\right) + O(\Delta x^2)$$

$$= f(x,t) + \frac{\Delta t}{2} f_t(x,t) + O(\Delta t^2)$$

$$+ \frac{\Delta x}{2}\left(f_x(x,t) + \frac{\Delta t}{2} f_{xt}(x,t) + O(\Delta t^2)\right)$$

$$= f(x,t) + \frac{\Delta t}{2} f_t(x,t) + \frac{\Delta x}{2} f_x(x,t) + O(\Delta t^2) + O(\Delta x \Delta t) + O(\Delta x^2)$$

Substituting this expression into $\boxed{b}$, and using $\Delta t \sim \Delta x$ from $\boxed{a}$

$$x(t+\Delta t) \cong x(t) + \Delta t\, f(x,t) + \frac{\Delta t^2}{2} f_t(x,t) + \frac{\Delta t\, \Delta x}{2} f_x(x,t) + O(\Delta t^3)$$

$$= x(t) + \Delta t\, f(x,t) + \frac{\Delta t^2}{2}\left(f_t(x,t) + f_x(x,t)\, f(x,t)\right) + O(\Delta t^3)$$

$$= x(t) + \Delta t\, \dot{x}(t) + \frac{\Delta t^2}{2}\ddot{x}(t) + O(\Delta t^3)$$

### Taylor Series expansion of $x(t+\Delta t)$ about $t$:

$$\underbrace{x(t+\Delta t) = x(t) + \Delta t\, \dot{x}(t) + \frac{\Delta t^2}{2}\ddot{x}(t)} + \underbrace{O(\Delta t^3)}$$

Midpoint Method is equivalent to T.S. expansion up to the quadratic term.

these terms differ between Midpoint Method & the T.S. expansion hence Midpoint Method makes an $O(\Delta t^3)$ error in each time step.

The error in one step of Mid. Method is $O(\Delta t^3)$.

To reach some time $T$ in the future, we must take $\frac{T}{\Delta t}$ steps.

Therefore, the accumulated error is

$$\frac{T}{\Delta t} \cdot O(\Delta t^3) = O(\Delta t^2)$$

Thus the Midpoint Method is said to be second order accurate.

# Stability of Euler's Method

## Euler's Method

$$x(t+\Delta t) \cong x(t) + \Delta t\, f(x,t)$$

To analyze the stability of a method, we study its behavior for a particular choice of $f$

$$f(x,t) = -kx \quad , \quad k > 0$$

So our diff. eq. is

$$\dot{x}(t) = -k\,x(t)$$

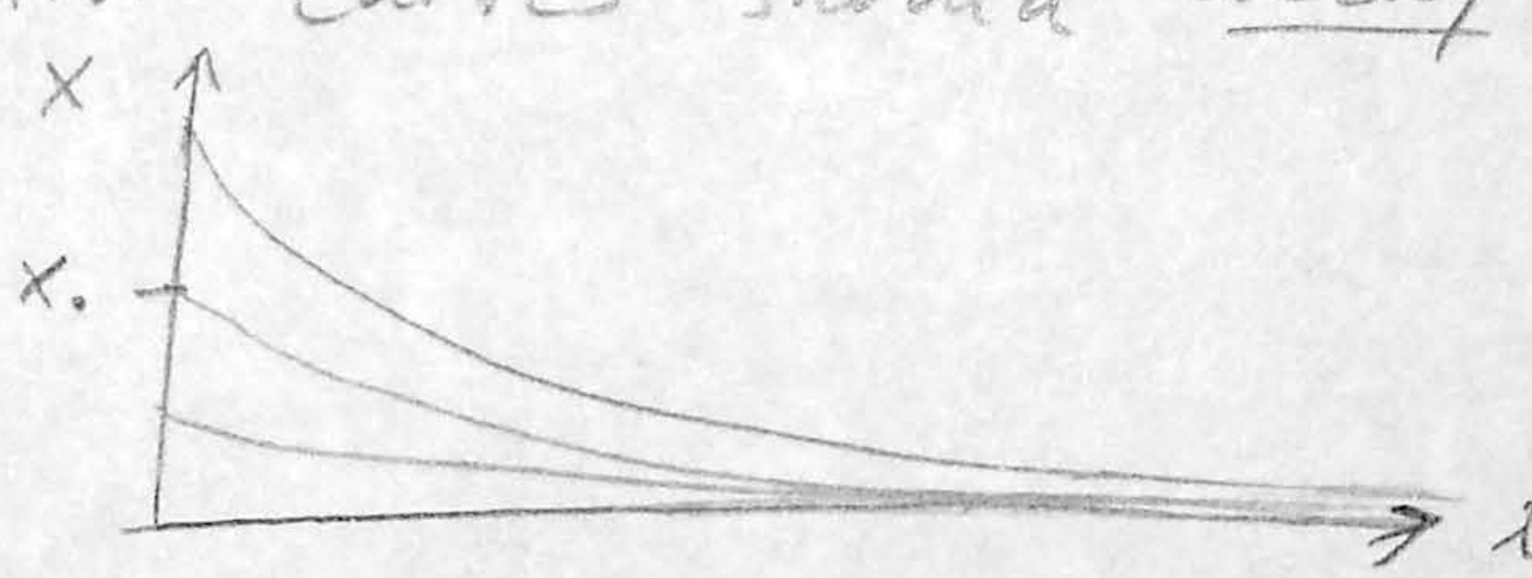
Given the initial value

$$x(t_0) = x_0$$

the exact solution to this I.V.P. is

$$x(t) = x_0\, e^{-k(t-t_0)}$$

so solution curves should decay as in the figure




Plugging this choice of $f$ into Euler's Method, we get

$$\begin{aligned} x(t+\Delta t) &\cong x(t) - \Delta t\, k\, x(t) \\ &= (1-\Delta t\, k)\, x(t) \end{aligned}$$

or

$$x(t+\Delta t) \cong A(\Delta t)\, x(t), \text{ where}$$

$$A(\Delta t) = (1-\Delta t\, k)$$

$A(\Delta t)$ is called the <u>amplification</u> <u>factor</u> because in each Euler step, we simply multiply (amplify) the previous value of $x$ by $A$. Note that $A$ depends on our choice of time step.

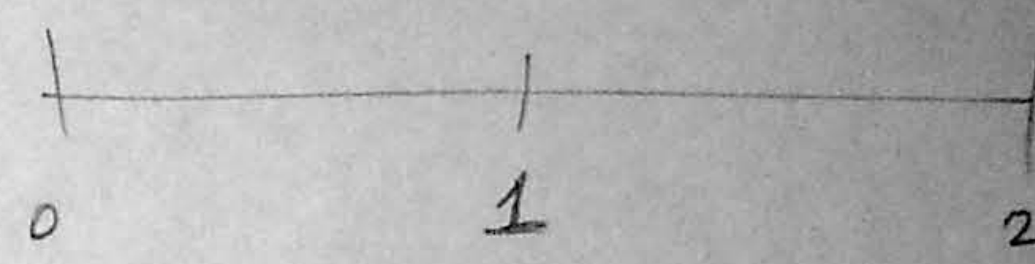
Since the true solution decays, in order to have any hope of faithfully approximating the true solution we should have

$$|A(\Delta t)| < 1$$

so that the numerical solution decays, too.

Therefore, we must choose $\Delta t$ so that

$$|1 - k\Delta t| < 1$$




or so that $0 < k\Delta t < 2$

This gives us a <u>time step restriction</u> of

$$\boxed{\Delta t < \frac{2}{k}}$$

Note that the larger $k$ is, the smaller time step we are forced to take.