23. Consider a ray with endpoint $\mathbf{a}$ and a normalized direction $\mathbf{u}$,

$$\mathbf{p}(t) = \mathbf{a} + t\mathbf{u}, \quad t \geq 0, \tag{1}$$

and a sphere of radius r, centered at the origin. The implicit equation for the sphere is given as follows:

$$f(\mathbf{p}) = \mathbf{p} \cdot \mathbf{p} - r^2 = 0 \tag{2}$$

(a) Describe geometrically the ways in which the ray can intersect/not intersect with the sphere. I.e., when is there exactly one intersection, when are there two intersections, and when are there no intersections?

Answer

We enumerate the ways in which the line containing the ray can intersect the sphere, and for each case how the ray itself ($t \geq 0$) can intersect the sphere. These cases are illustrated below.

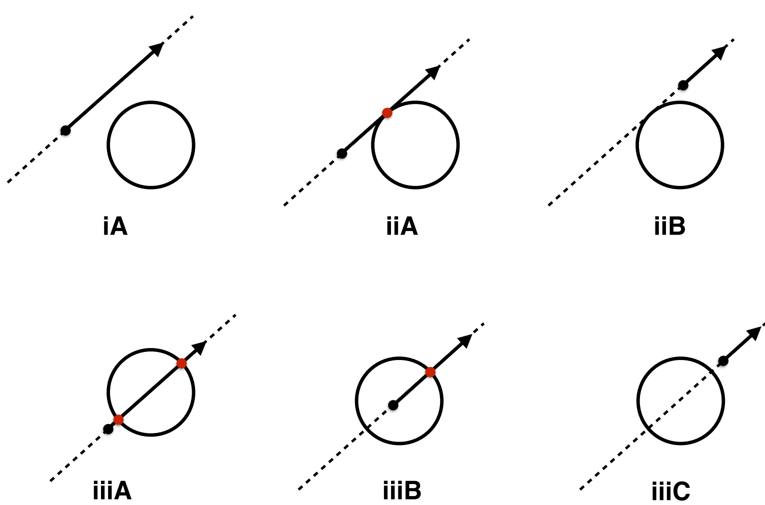
i. 0 intersection points of line and sphere
   - A. the ray also also does <u>not</u> intersect with the sphere

ii. 1 intersection point of line and sphere
   - A. the ray contains the intersection point
   - B. the ray does <u>not</u> contain the intersection point

iii. 2 intersection points of line and sphere
   - A. the ray contains point intersection points
   - B. the contains exactly one intersection point
   - C. the ray does <u>not</u> contain either intersection point

In summary,

- 0 intersection points: iA, iiB, and iiiC
- 1 intersection point: iiA and iiiB
- 2 intersection points: iiiA



(b) Find an expression for t where the intersection occurs by plugging eq. (1) into eq. (2) and solving for t. How can this expression to be used to distinguish the three cases described in part (a)? *Hint (Quadratic formula): Solutions to* $ax^2 + bx + c = 0$ *are* $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Answer

Plugging eq. (1) into eq. (2), we get

$$f(\mathbf{p}(t)) = (\mathbf{a} + t\mathbf{u}) \cdot (\mathbf{a} + t\mathbf{u}) - r^2 = 0.$$

This gives

$$\begin{aligned}(\mathbf{a} + t\mathbf{u}) \cdot (\mathbf{a} + t\mathbf{u}) - r^2 &= 0 \\ \mathbf{a} \cdot \mathbf{a} + 2t\mathbf{a} \cdot \mathbf{u} + t^2\mathbf{u} \cdot \mathbf{u} - r^2 &= 0 \\ (\mathbf{u} \cdot \mathbf{u})\, t^2 + (2\mathbf{a} \cdot \mathbf{u})\, t + \left(\mathbf{a} \cdot \mathbf{a} - r^2\right) &= 0.\end{aligned}$$

We know $\mathbf{u} \cdot \mathbf{u} = 1$ since the direction vector $\mathbf{u}$ of the ray is normalized. So this simplifies to

$$t^2 + (2\mathbf{a} \cdot \mathbf{u})\, t + \left(\mathbf{a} \cdot \mathbf{a} - r^2\right) = 0.$$

Using the quadratic formula given in the hint, we solve for t:

$$t = \frac{-2\mathbf{a} \cdot \mathbf{u} \pm \sqrt{4(\mathbf{a} \cdot \mathbf{u})^2 - 4(\mathbf{a} \cdot \mathbf{a} - r^2)}}{2}$$

Simplifying further, the expression for t is

$$\boxed{t = -\mathbf{a} \cdot \mathbf{u} \pm \sqrt{(\mathbf{a} \cdot \mathbf{u})^2 - (\mathbf{a} \cdot \mathbf{a} - r^2)}}$$

We will label the two roots as

$$\begin{aligned}t1 &= -\mathbf{a} \cdot \mathbf{u} - \sqrt{(\mathbf{a} \cdot \mathbf{u})^2 - (\mathbf{a} \cdot \mathbf{a} - r^2)}, \\ t2 &= -\mathbf{a} \cdot \mathbf{u} + \sqrt{(\mathbf{a} \cdot \mathbf{u})^2 - (\mathbf{a} \cdot \mathbf{a} - r^2)}.\end{aligned}$$

The different intersection cases can be distinguished by the value of the discriminant

$$d = (\mathbf{a} \cdot \mathbf{u})^2 - (\mathbf{a} \cdot \mathbf{a} - r^2),$$

and of $t1$ and $t2$. These are summarized in this table, with l = number of line/sphere intersections, r = number of ray/sphere intersections

d	$t1$	$t2$	fig	l	r
$d < 0$	$t1$ complex	$t2$ complex	iA	0	0
$d = 0$	$t1 \geq 0$	$t2 \geq 0$	iiA	1	1
$d = 0$	$t1 \leq 0$	$t2 \leq 0$	iiB	1	0
$d > 0$	$t1 \geq 0$	$t2 \geq 0$	iiiA	2	2
$d > 0$	$t1 < 0$	$t2 \geq 0$	iiiB	2	1
$d > 0$	$t1 < 0$	$t2 < 0$	iiiC	2	0