

general rotations

Rotation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

X axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Y axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Z axis

Rotation about an arbitrary axis

Rotating about an axis by theta degrees

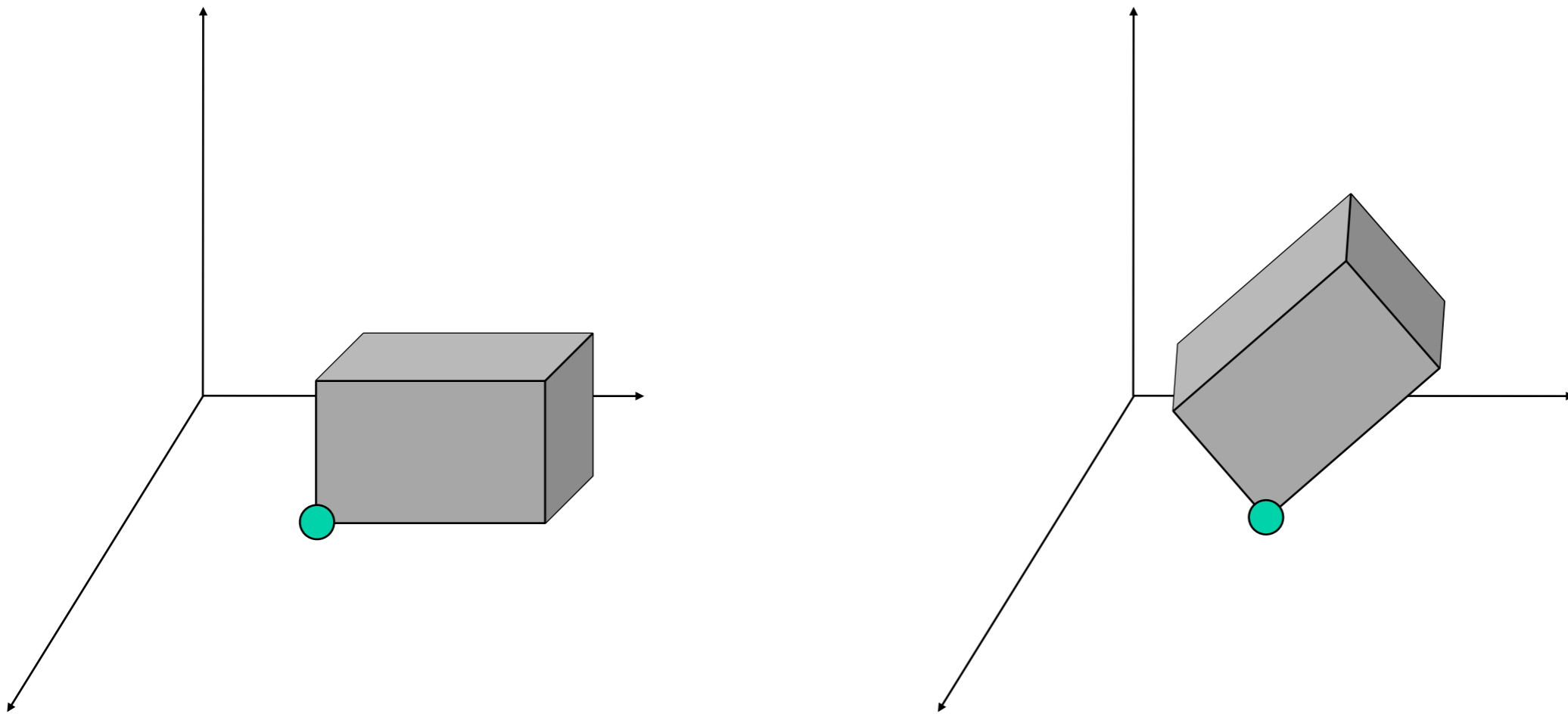
- Rotate about x to bring axis to xz plane
- Rotate about y to align axis with z -axis
- Rotate theta degrees about z
- Unrotate about y, unrotate about x

$$\mathbf{M} = \mathbf{R}_x^{-1} \mathbf{R}_y^{-1} \mathbf{R}_z(\theta) \mathbf{R}_y \mathbf{R}_x$$

- Can you determine the values of \mathbf{R}_x and \mathbf{R}_y ?

Composite Transformations

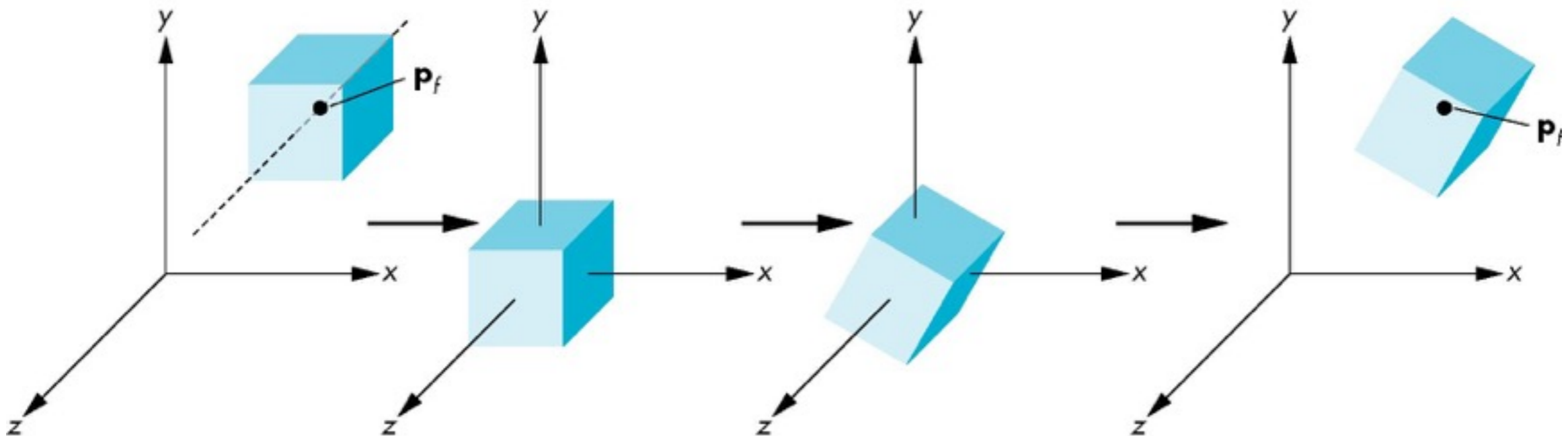
- Rotating about a fixed point
 - **basic** rotation alone will rotate about origin but we want:



Composite Transformations

- Rotating about a fixed point
- Move fixed point (p_x, p_y, p_z) to origin
- Rotate by desired amount
- Move fixed point back to original position

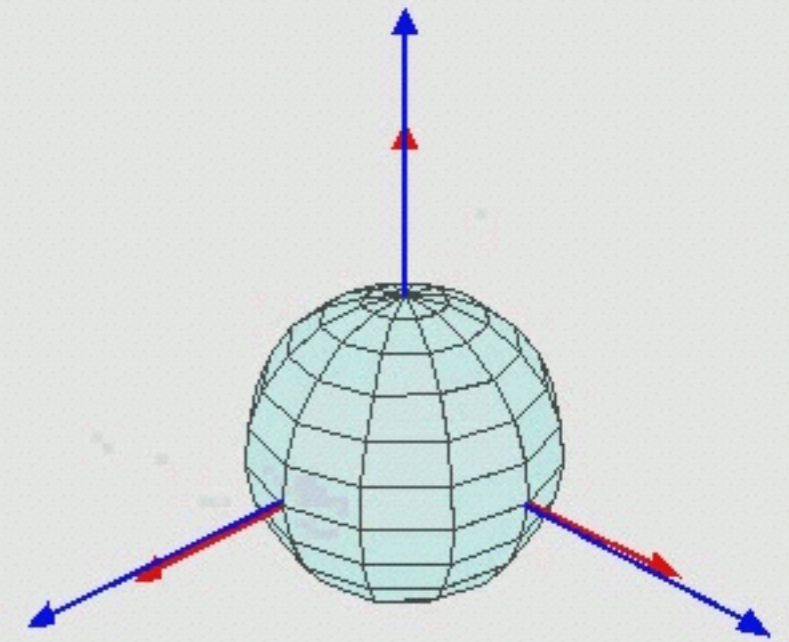
$$\mathbf{M} = \mathbf{T}(p_x, p_y, p_z) \mathbf{R}_z(\theta) \mathbf{T}(-p_x, -p_y, -p_z)$$



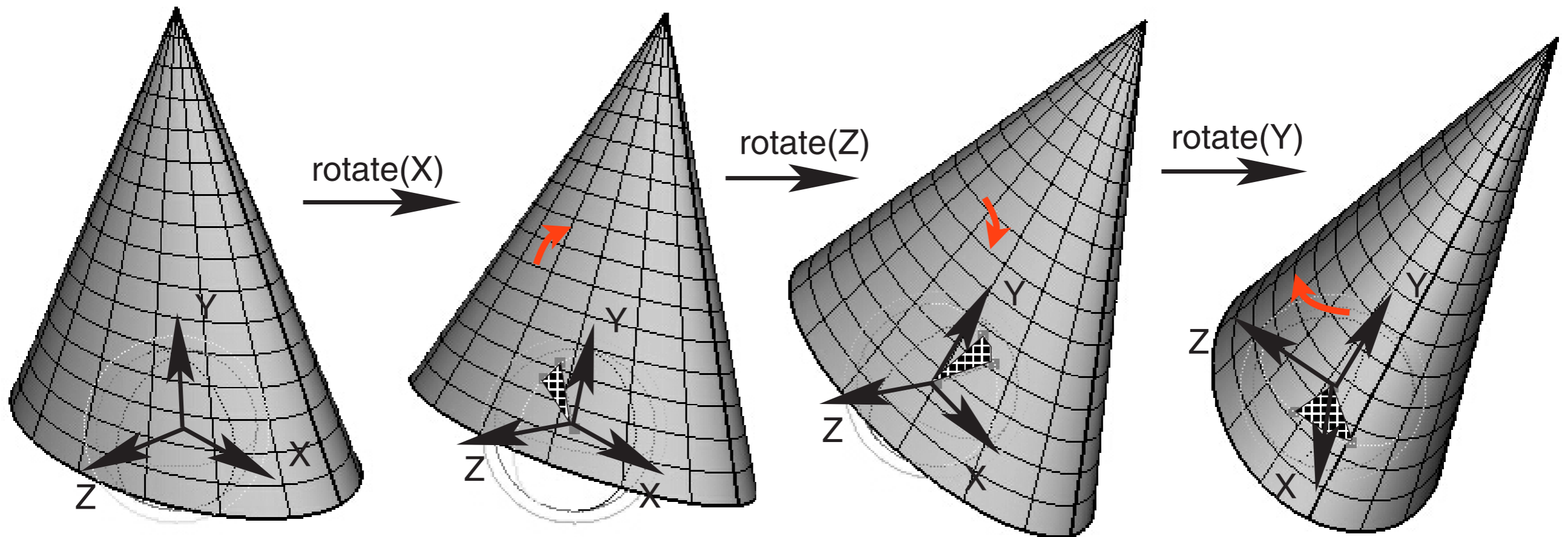
euler angles

Euler Angles

3 Euler angles can be used to specify an arbitrary orientation

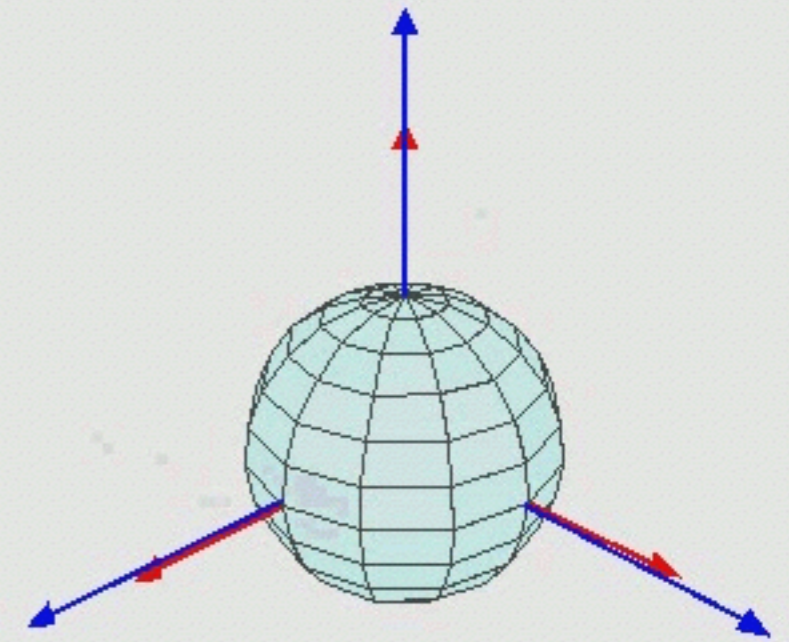


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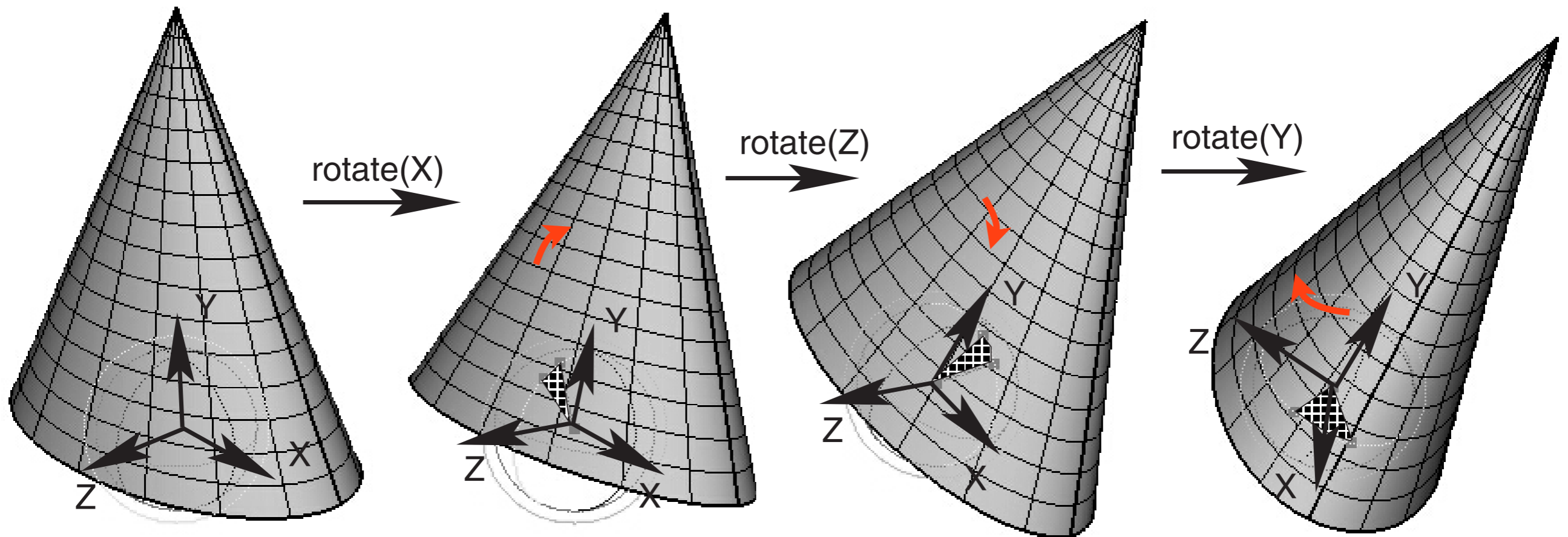


Euler Angles

3 Euler angles can be used to specify an arbitrary orientation

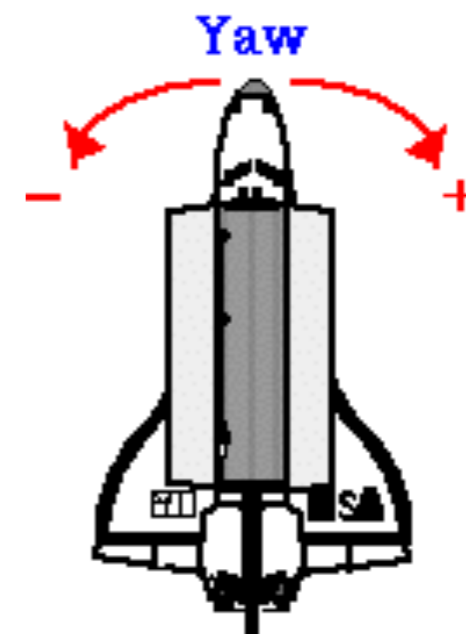
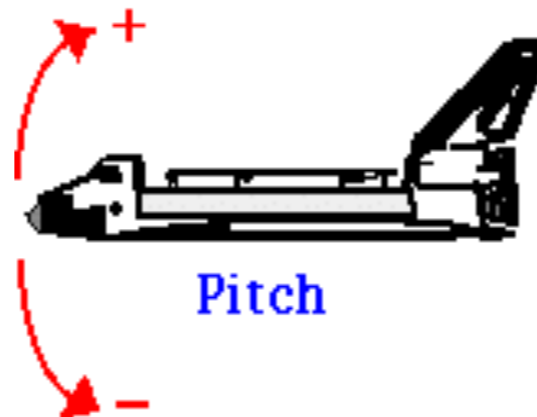
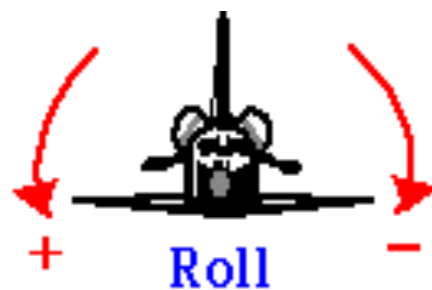


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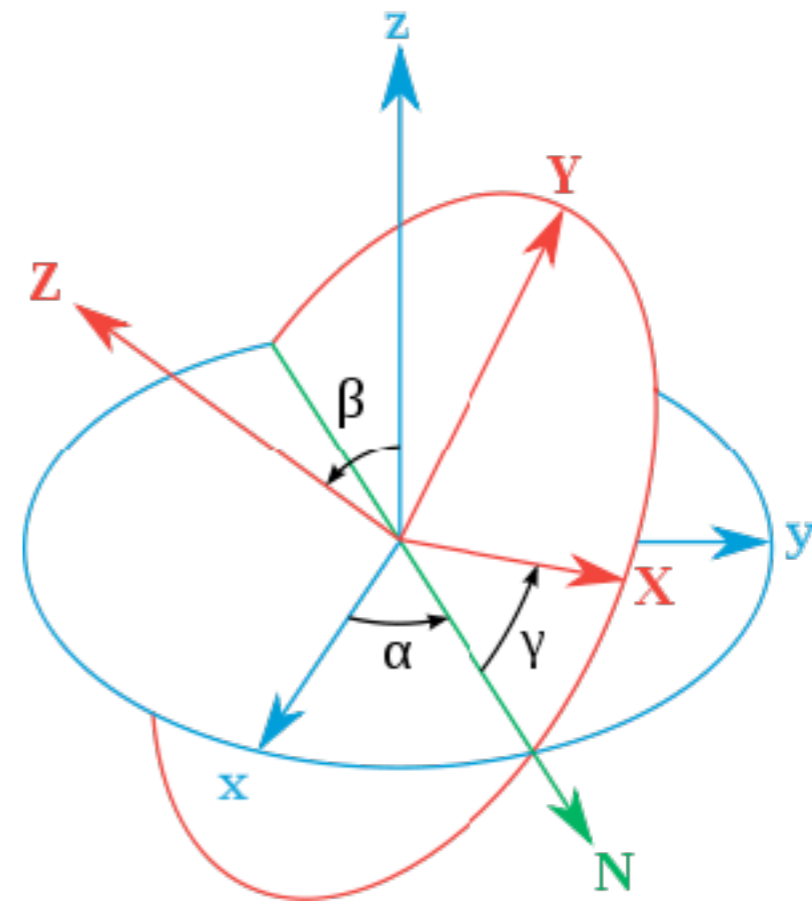
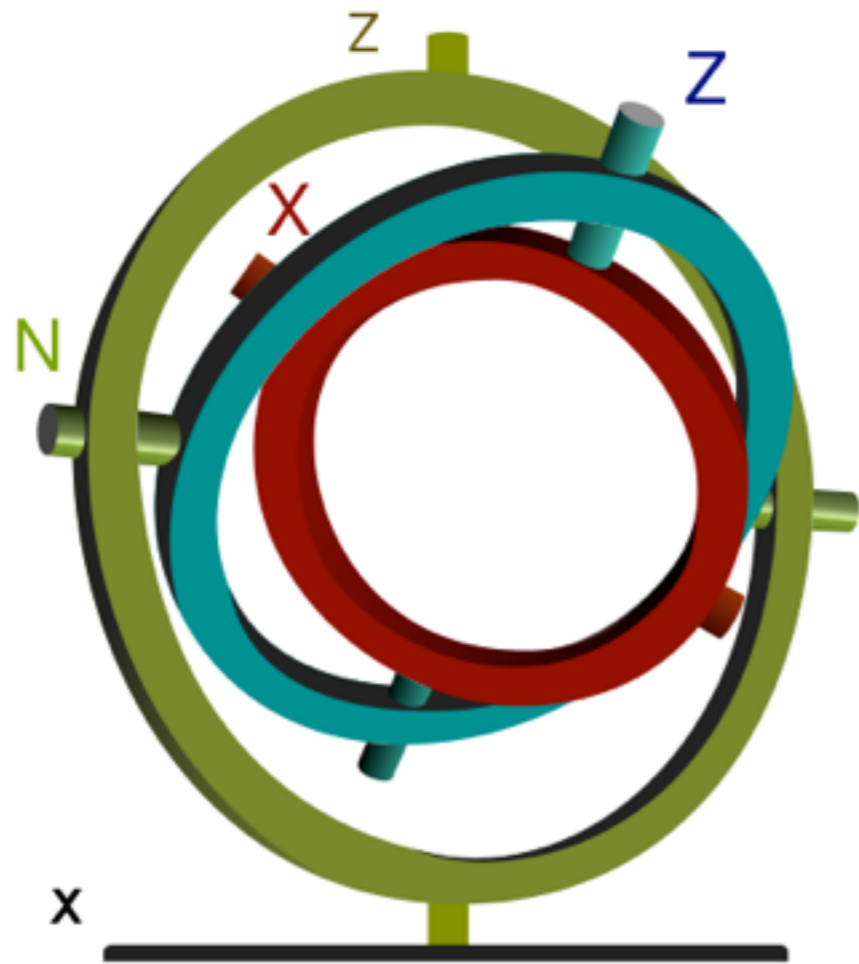


Euler Angles

- A general rotation is a combination of three elementary rotations:
 - around the x-axis (roll)
 - around the y-axis (pitch)
 - around the z-axis (yaw)

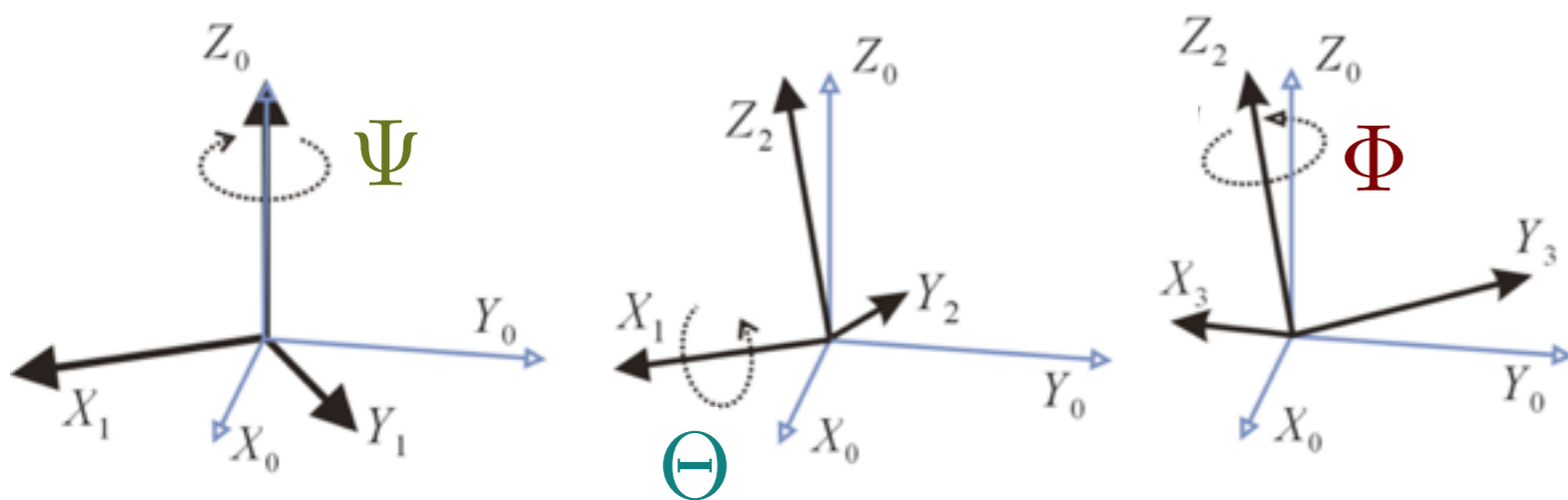


Gimbal and Euler Angles

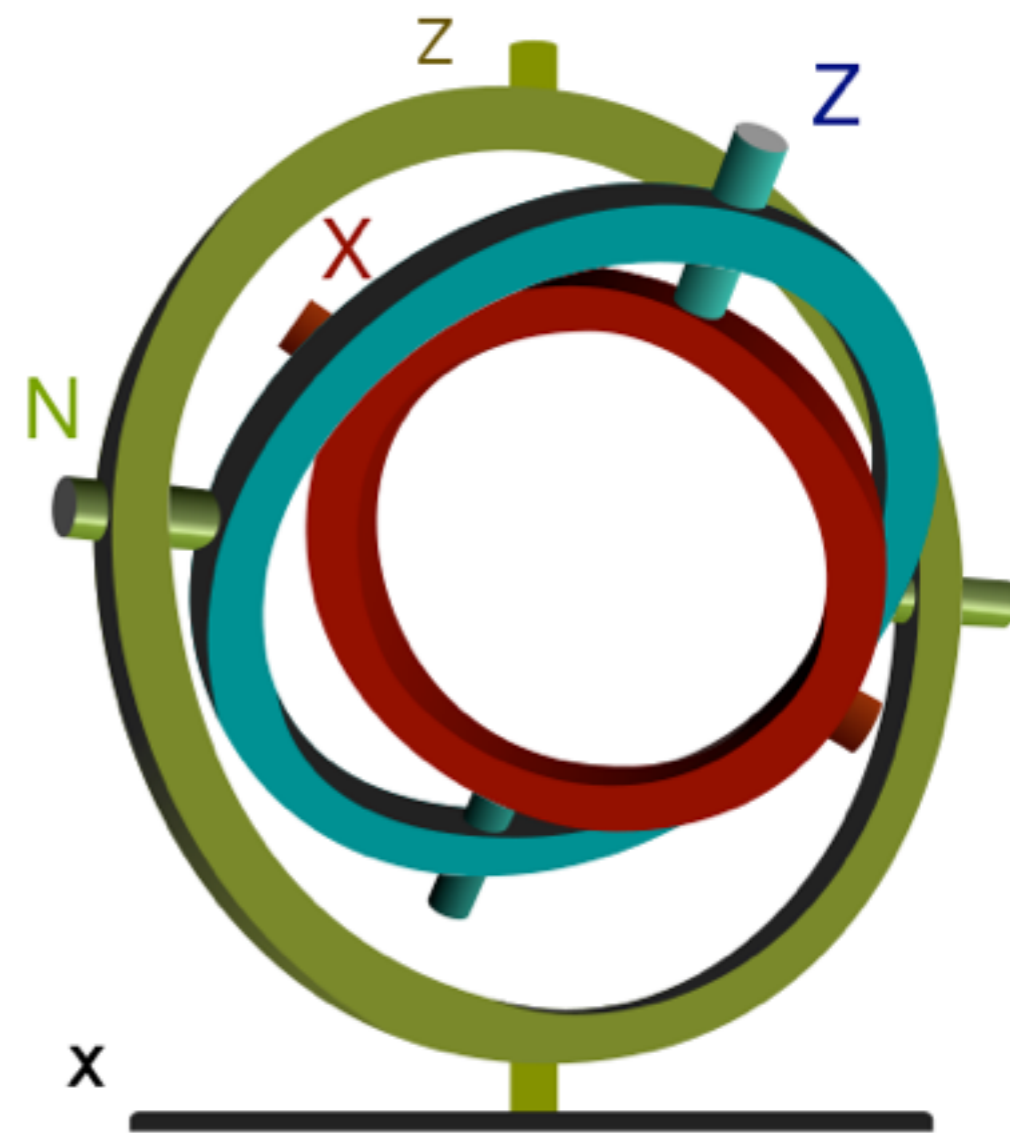
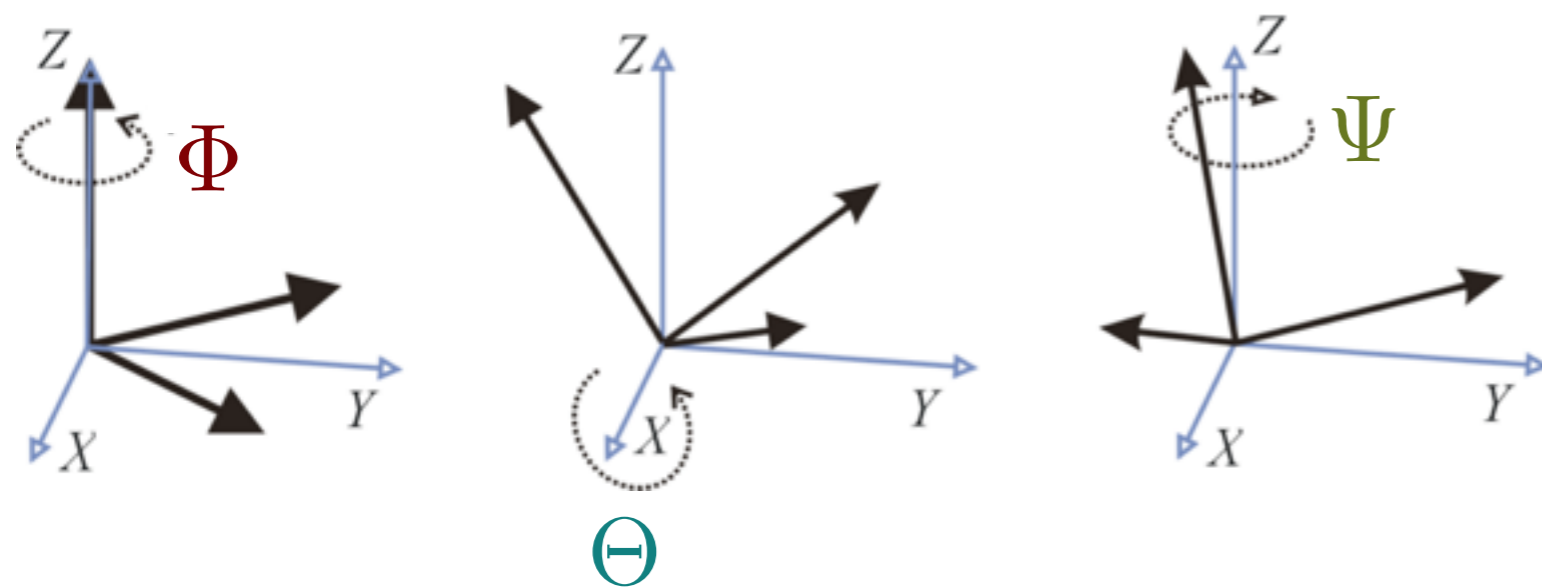


Z-X'-Z''

intrinsic



extrinsic



Euler Angles and Rotation Matrices

$$\text{x-roll}(\theta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 & 0 \\ 0 & \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{y-pitch}(\theta_2) = \begin{pmatrix} \cos \theta_2 & 0 & \sin \theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{z-yaw}(\theta_3) = \begin{pmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} c_2 c_3 & -c_2 s_3 & s_2 & 0 \\ s_1 s_2 c_3 + c_1 s_3 & -s_1 s_2 s_3 + c_1 c_3 & -s_1 c_2 & 0 \\ -c_1 s_2 c_3 + s_1 s_3 & c_1 s_2 s_3 + s_1 c_3 & c_1 c_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

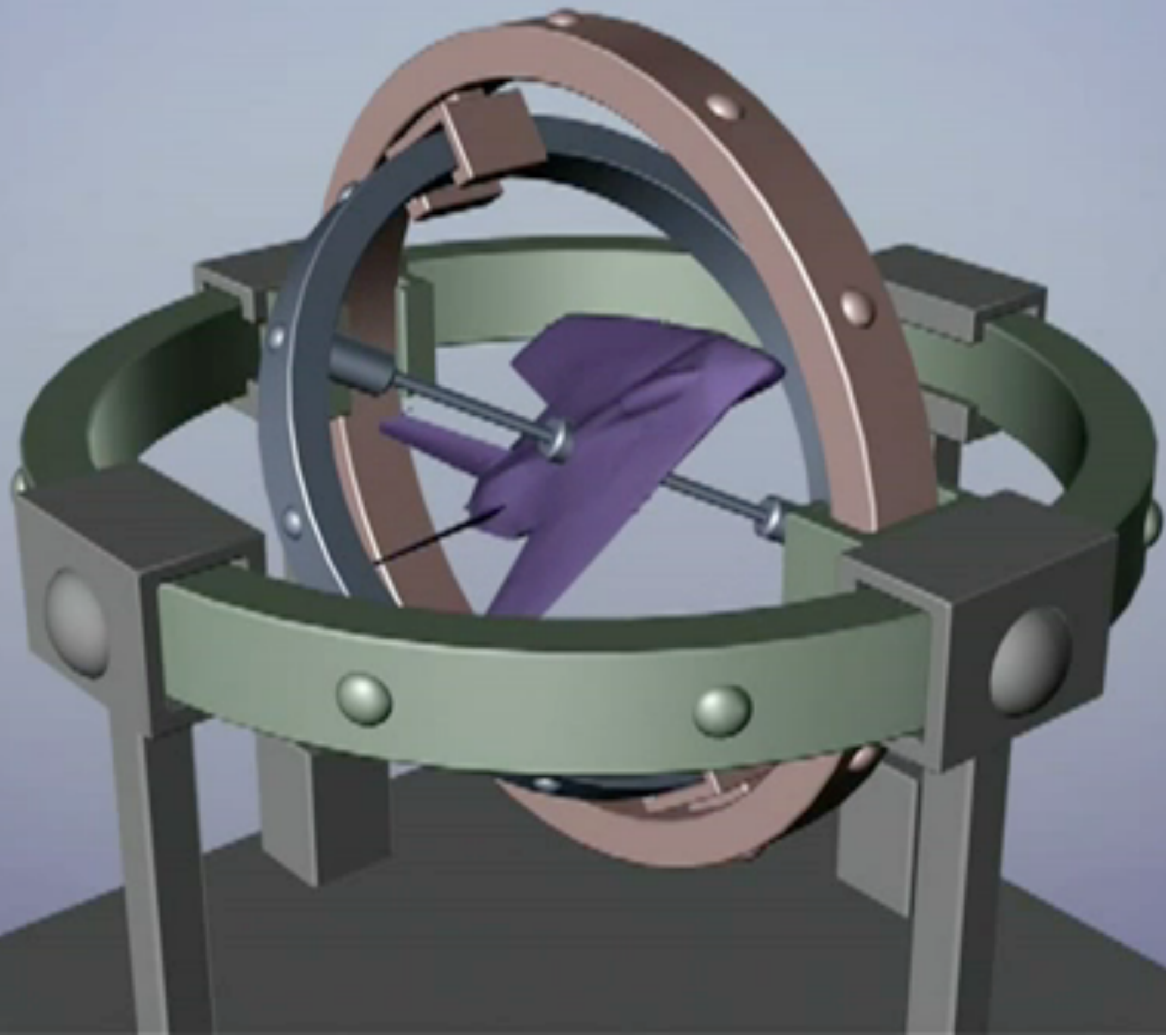
Interpolating Rotations

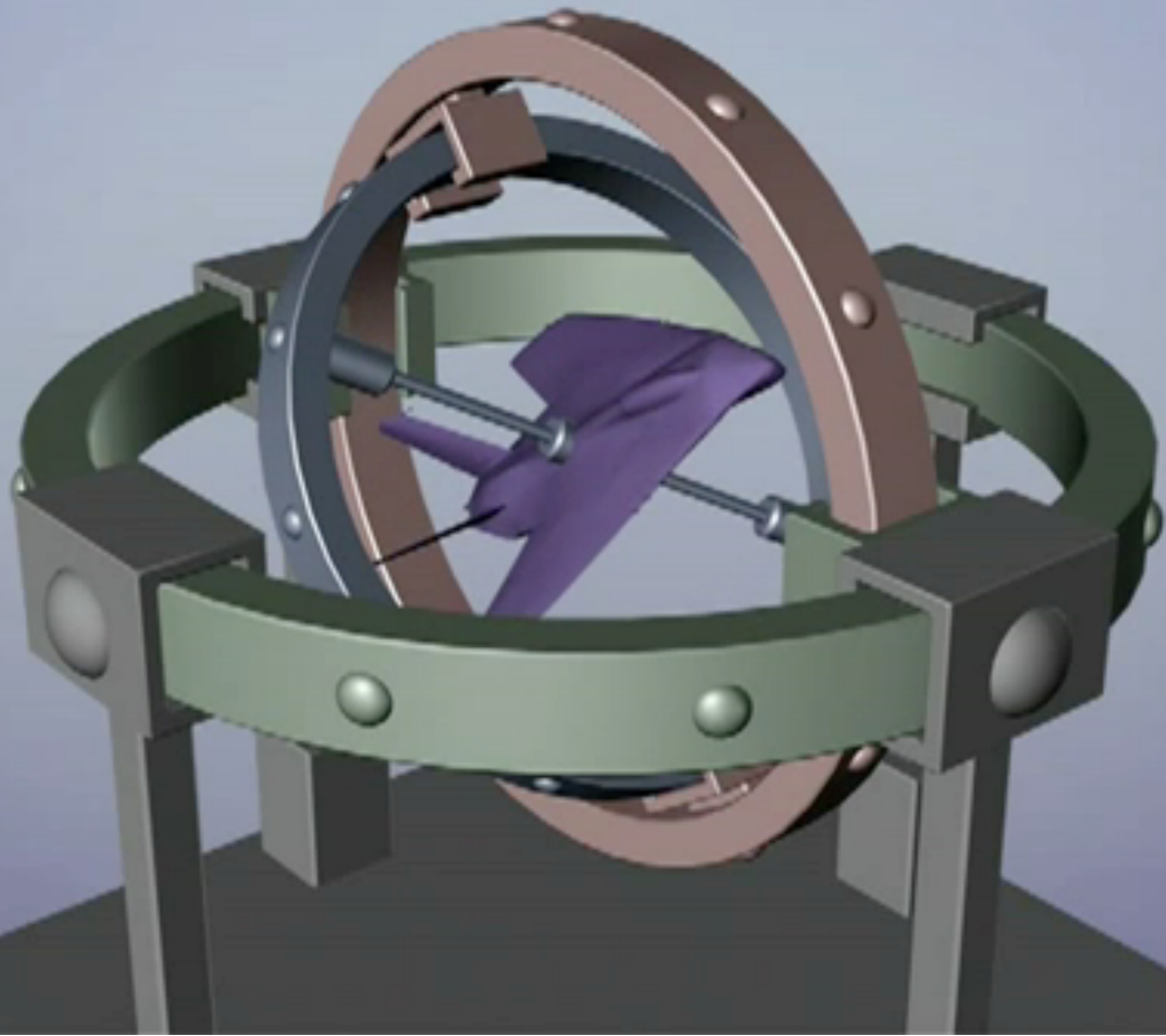
- naive element-by-element interpolation of rotation matrices does not work! E.g.,

$$\frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

90 CW 90 CCW

- We can interpolate Euler angles, but there is a problem known as **Gimbal Lock**





quaternions

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication

$$i^2 = j^2 = k^2 = ijk = -1$$

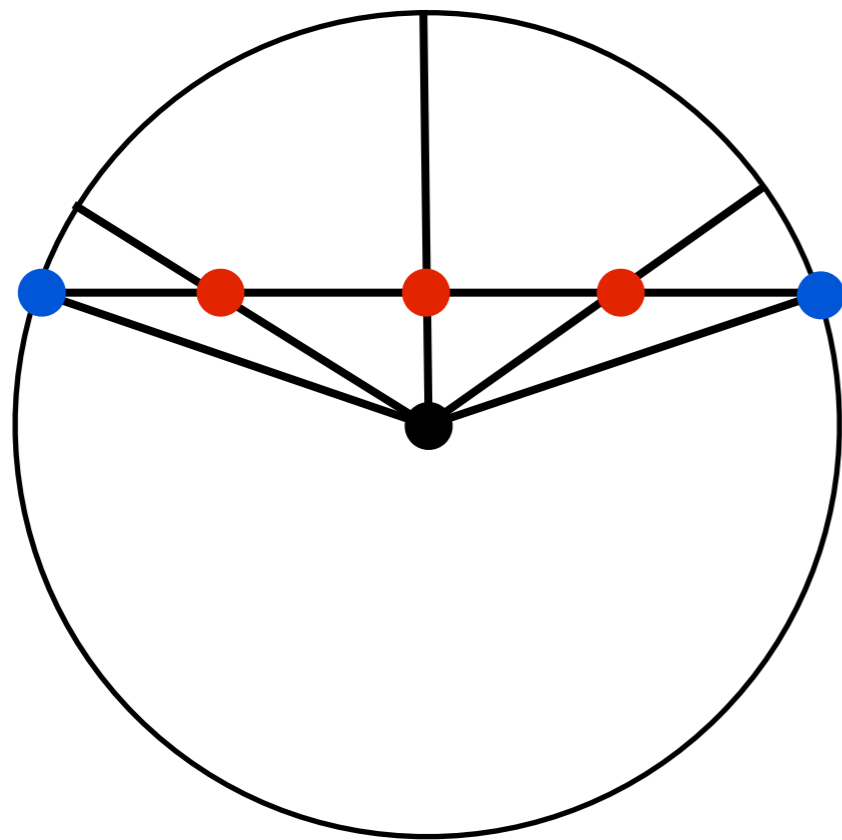
Carved on a stone of the bridge

Quaternions

- axis/angle representation
- interpolates smoothly
- easy to compose

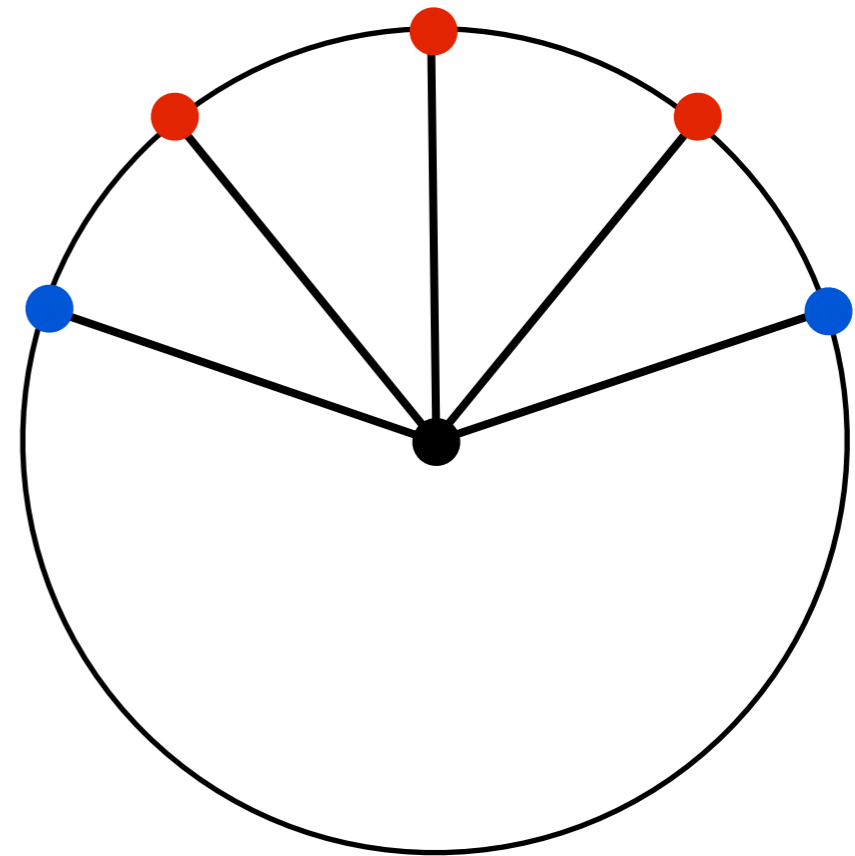
<whiteboard>

Quaternion Interpolation



linear

non-uniform speed



**spherical linear
(slerp)**

constant speed

Matrix form

$$\mathbf{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy + 2wz & 2xz - 2wy & 0 \\ 2xy - 2wz & 1 - 2x^2 - 2z^2 & 2yz + 2wx & 0 \\ 2xz + 2wy & 2yz - 2wx & 1 - 2x^2 - 2y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations in Reality

- It's easiest to express rotations in Euler angles or Axis/angle
- We can convert to/from any of these representations
- Choose the best representation for the task
 - input: Euler angles
 - interpolation: quaternions
 - composing rotations: quaternions, orientation matrix