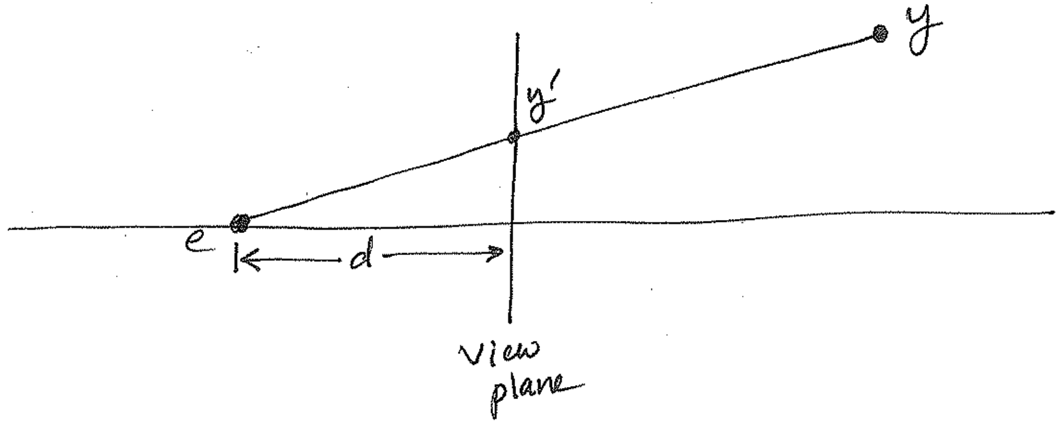
Notes on Perspective Transformation

① Simple perspective projection



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix} \Rightarrow \begin{cases} x' = \frac{d}{z} x \\ y' = \frac{d}{z} y \\ z' = \frac{d}{z} z = d \end{cases}$$

- this achieves a simple perspective projection onto the viewplane $z = d$
- But we have lost all information about the original z coordinate!

② Perspective - normalization transformation

- To preserve z information, we will derive a transformation which yields perspective-projected x and y values but is non-singular,
- start with the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & 1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \alpha z + \beta \\ z/d \end{bmatrix} \Rightarrow \begin{aligned} x' &= \frac{d}{z} x \\ y' &= \frac{d}{z} y \\ z' &= \frac{d}{z}(\alpha z + \beta) \\ &= d\alpha + \frac{d\beta}{z} \end{aligned}$$

- To find α and β, we'll impose the two conditions

$$\left.\begin{aligned} (1) \quad n &= d\alpha + \frac{d\beta}{n} \\ (2) \quad f &= d\alpha + \frac{d\beta}{f} \end{aligned}\right]$$

i.e., the z values of the near and far planes are preserved under the transformation

- We solve the above two equations :

$$(1)-(2) \Rightarrow n - f = d\beta\left(\frac{1}{n} - \frac{1}{f}\right) = d\beta\left(\frac{f-n}{nf}\right) \Rightarrow \beta = \frac{nf(n-f)}{d(f-n)} = \boxed{-\frac{nf}{d} = \beta}$$

Substituting for β in (1),

$$n = d\alpha + \frac{d}{n}\beta = d\alpha + \frac{\cancel{d}}{\cancel{n}}\left(-\frac{\cancel{n}f}{\cancel{d}}\right) = d\alpha - f \Rightarrow \boxed{\alpha = \left(\frac{n+f}{d}\right)}$$

- Since e is at the origin, $d = n$.

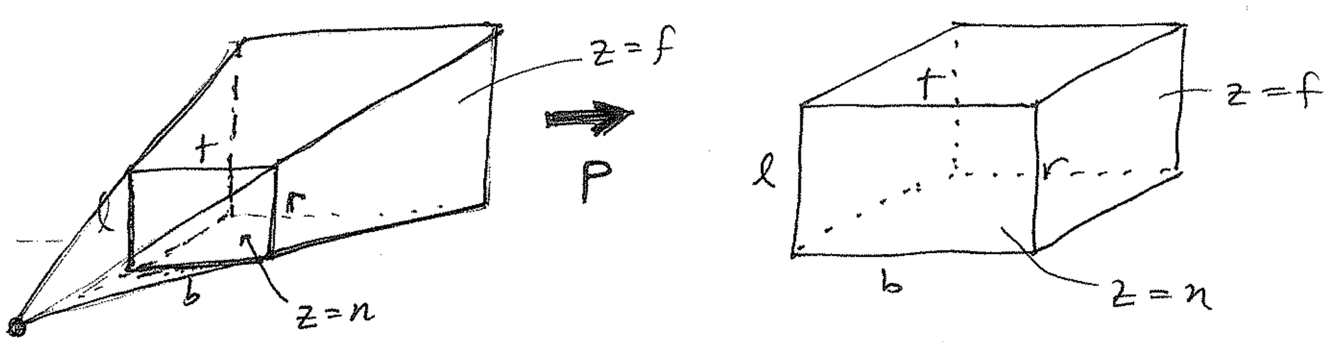
- substituting these results into our transformation, we get

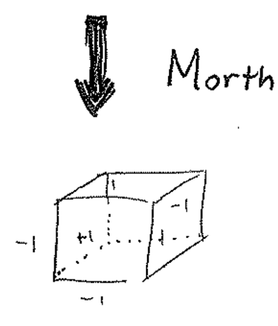
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{n+f}{n} & -f \\ 0 & 0 & 1/n & 0 \end{bmatrix}$$

- to avoid the division, we multiply through by n (recalling that $\alpha(x,y,z,w)^T$ + $(x,y,z,w)^T$ give the same point)

$$P = \begin{bmatrix} n & & & \\ & n & & \\ & & n+f & -nf \\ & & 1 & 0 \end{bmatrix}$$

- this has achieved the transformation





- Finally, we combine this with the orthographic transformation, M_{orth}, to get to the canonical viewing volume, so $\boxed{M_{per} = M_{orth} P}$

- recall that

$$M_{orth} = \begin{bmatrix} \frac{2}{r-l} & & & -\frac{(l+r)}{(r-l)} \\ & \frac{2}{t-b} & & -\frac{(b+t)}{t-b} \\ & & \frac{2}{n-f} & -\frac{(f+n)}{n-f} \\ & & & 1 \end{bmatrix}$$

- so

$$M_{per} = M_{orth} P = \begin{bmatrix} \frac{2n}{r-l} & & -\frac{(l+r)}{r-l} & \\ & \frac{2n}{t-b} & -\frac{(b+t)}{t-b} & \\ & & \frac{n+f}{n-f} & -\frac{2nf}{n-f} \\ & & 1 & 0 \end{bmatrix}$$

- Note : OpenGL ① assumes a right-handed system before proj. and a left-handed system after proj.
② assumes that glOrtho and glFrustum will be passed near $= -n$ and far $= -f$

① and ② are equivalent to negating the 3rd row and 3rd col of Morth or Mper. Hence,

$$M_{orth}^{openGL} = \begin{bmatrix} \frac{2}{r-l} & & & -\frac{(l+r)}{r-l} \\ & \frac{2}{t-b} & & -\frac{(b+t)}{t-b} \\ & & \frac{2}{n-f} & \frac{f+n}{n-f} \\ & & & 1 \end{bmatrix} ; \quad M_{per}^{openGL} = \begin{bmatrix} \frac{2n}{r-l} & & \frac{l+r}{r-l} & \\ & \frac{2n}{t-b} & \frac{b+t}{t-b} & \\ & & \frac{n+f}{n-f} & \frac{2nf}{n-f} \\ & & -1 & \end{bmatrix}$$

Note that the (3,3) element gets multiplied by $(-1)(-1) = 1$, so stays the same.