10.1 Curves & Surfaces.

[Source: Angel & Shreiner Chapter 10.]

① Explicit Representation — (limited)

gives value of <u>dependent variable</u>

in terms of <u>independent variable</u>.

E.g.

dependent $\quad y = f(x)$, or $\quad$ independent

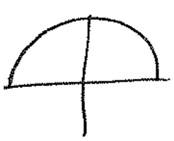
$$x = f(y)$$

(*) $\quad y = mx + b$

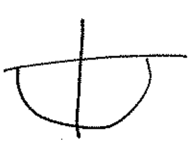
Form is not guaranteed to exist for a given curve.

E.g., vertical line in (*).

<u>Example</u> Circle.



$$y = (r^2 - x^2)^{1/2}$$ half the circle.



$$y = -(r^2 - x^2)^{1/2}$$ other half.

restriction $\quad 0 \le |x| \le r$

In 3D, explicit curve representation:

$$y = f(x)$$
$$z = g(x)$$

Surface requires 2 indep. variables:

$$z = f(x,y)$$

Curve or surface may not have explicit representation

Eg. curve

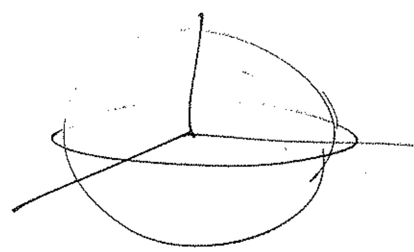
$$\left.\begin{aligned} y &= ax+b \\ z &= cx+d \end{aligned}\right\} \text{ can't describe line in constant } x\text{-plane.}$$

Eg. surface

sphere

$$z = f(x,y)$$

given x, y give 0, 1, or 2 points on sphere.



Implicit Representations

limitation is obtaining points on the curve or surface.

$\boxed{\text{2D implicit curve}}$

$$f(x,y) = 0.$$

line $\quad ax + by + c = 0$

Circle $\quad x^2 + y^2 - r^2 = 0.$

f is a "testing" or "membership" function but it is not necessarily easy to find a y satisfying equation, given x.

$\boxed{\text{3D implicit}}$

$\boxed{\text{surface}}$ $\quad f(x,y,z) = 0.$

e.g. plane $\quad ax + by + cz + d = 0.$

e.g. sphere $\quad x^2 + y^2 + z^2 - r^2 = 0$

$\boxed{\text{curve}}$ $\quad$ intersection of 2 surfaces.

$$f(x,y,z) = 0$$

$$g(x,y,z) = 0.$$

Algebraic surfaces:

$$f(x, y, z) = 0$$

f is a sum of polynomials in x, y, z

e.g. quadric surfaces

each term in f has degree ≤ 2.

intersections w/ lines has at most two intersection points.

(useful ~~fa~~ fact for rendering).

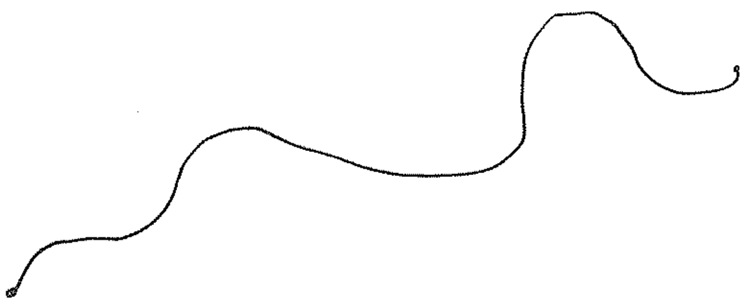
Parametric Form

parameter, u

$$\left.\begin{aligned} x &= x(u) \\ y &= y(u) \\ z &= z(u) \end{aligned}\right\} \text{ curve in 3 dimensions}$$

- similar in 2D

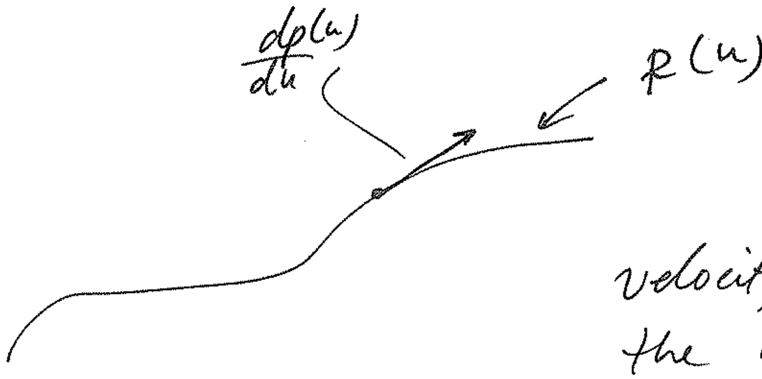
$$\left.\begin{aligned} x &= x(u) \\ y &= y(u) \end{aligned}\right\} \text{ curve in 2 dimensions.}$$



locus of points drawn as u varies.

$$p(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix}$$

Tangent Vector



velocity with which the curve is traced out.

Points in the direction tangent to the curve.

$$\frac{dp}{du} = \begin{pmatrix} \frac{dx}{du} \\ \frac{dy}{du} \\ \frac{dz}{du} \end{pmatrix}$$

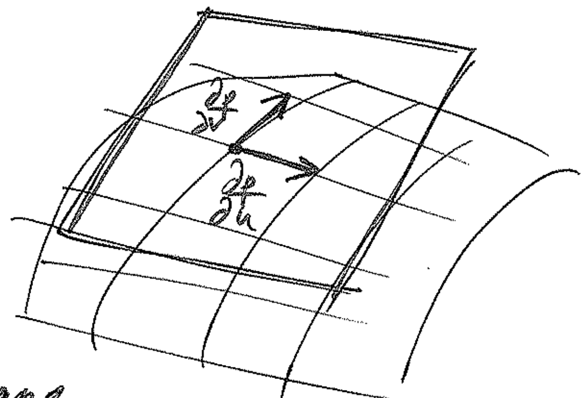
Parametric Surfaces

- require 2 parameters, u, v

$$\left.\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}\right\} \text{ surface in 3D.}$$

$$\cancel{p(u,v)} \quad p(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

$$\frac{\partial p}{\partial u} = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}, \quad \frac{\partial p}{\partial v} = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}$$

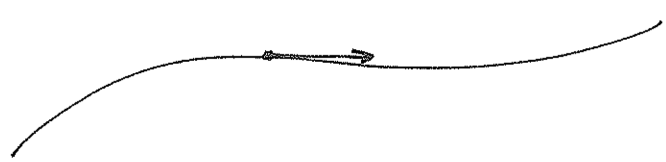


tangent vectors define tangent plane.

$$\underline{N} = \frac{\partial p}{\partial u} \times \frac{\partial p}{\partial v}$$

cross product gives normal.

E.g., Frenet Frame



tangent
normal
binormal.

parametric polynomial Curves

- non-unique.
- widely used in computer graphics

$$\underset{\sim}{p}(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix}$$

polynomials:

$$p(u) = \sum_{k=0}^{n} u^k \underset{\sim}{c}_k \quad , \quad c_k = \begin{pmatrix} c_{xk} \\ c_{yk} \\ c_{zk} \end{pmatrix}$$

$$\boxed{p(u) = \sum_{k=0}^{n} u^k \begin{pmatrix} c_{xk} \\ c_{yk} \\ c_{zk} \end{pmatrix}}$$

c_k independent x, y, z components.

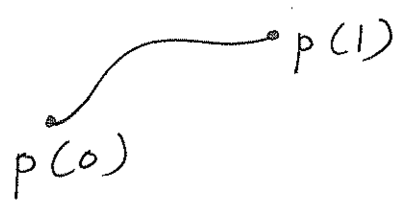
$n+1$ column matrices $\{c_k\}$ Coefficients of p.

The $\{c_k\}$ comprise $3(n+1)$ degrees of freedom in choosing p.

WLOG, often take $0 \le u, v \le 1$

curve segment

$0 \le u \le 1$



$u_{min} \le u \le u_{max}$

can be rescaled

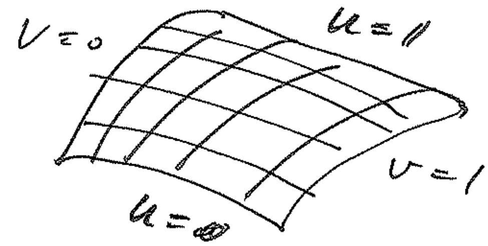
Parametric Polynomial Surfaces

$$p(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} = \left.\sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij}\, u^i v^j \right\} \text{ } 3(m+1)(n+1) \text{ coefficients.}$$

usually take $m = n$, $0 \leq u, v \leq 1$

↳ surface patch

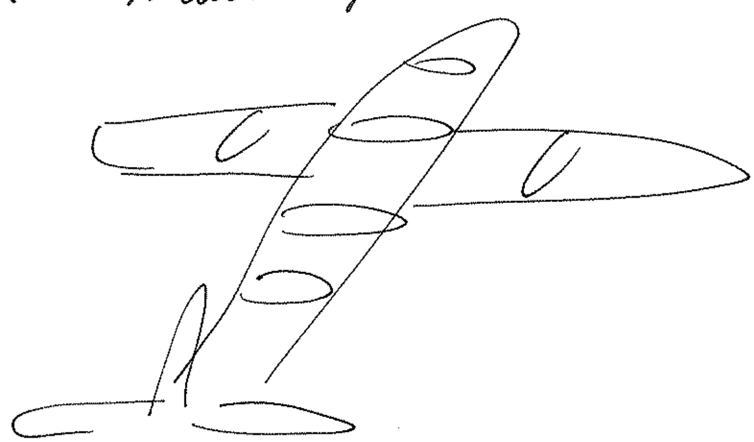
Note: if you hold u constant, get a parametric curve in v, + vice versa.



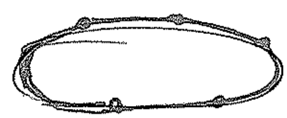
Design Considerations

- local control of shape
- smoothness + continuity
- ability to evaluate derivatives
- stability
- ease of rendering.

E.g. model airplane



cross section

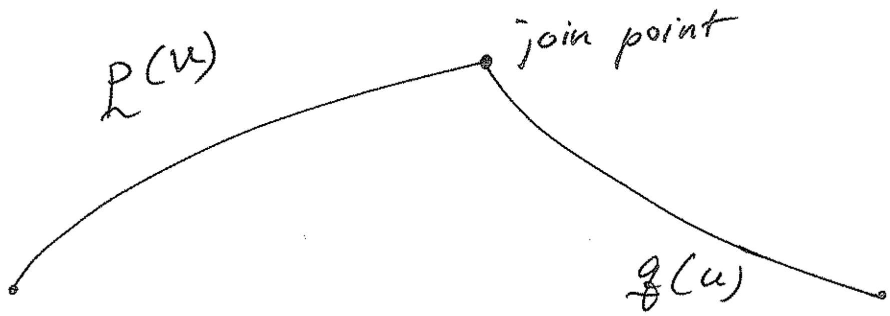


approximation out of number of wood strips.

join points.

For polynomial curves + surfaces, ~~smoot~~ segments are smooth + have all derivatives.

Complications may arise at join points.



— we want <u>**local control**</u>, so ~~des~~ we want to design each segment individually, + interactively

— we want stability: small changes in values of input parameters should cause only small changes in output.

Typically, consider data at a small ~~#~~ of <u>control points</u>.

Curve may interpolate control points

or

not directly interpolate, but come close to all

10.3 Parametric Cubic Polynomial Curves

High degree : + many parameters
- evaluating pts. on curve <u>costly</u>
- higher order – more oscillatory

use low degree polynomials, defined over <u>short</u> <u>interval</u>.

piecewise cubic polynomials

$$p(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = \sum_{k=0}^{3} c_k u^k$$

$$= \begin{pmatrix} c_{0x} \\ c_{0y} \\ c_{0z} \end{pmatrix} + \begin{pmatrix} c_{1x} \\ c_{1y} \\ c_{1z} \end{pmatrix} u + \begin{pmatrix} c_{2x} \\ c_{2y} \\ c_{2z} \end{pmatrix} u^2 + \begin{pmatrix} c_{3x} \\ c_{3y} \\ c_{3y} \end{pmatrix} u^3$$

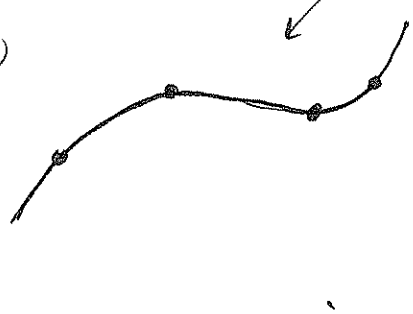
$$= \begin{pmatrix} | & | & | & | \\ c_0 & c_1 & c_2 & c_3 \\ | & | & | & | \end{pmatrix} \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix} = C\underset{\sim}{u}$$

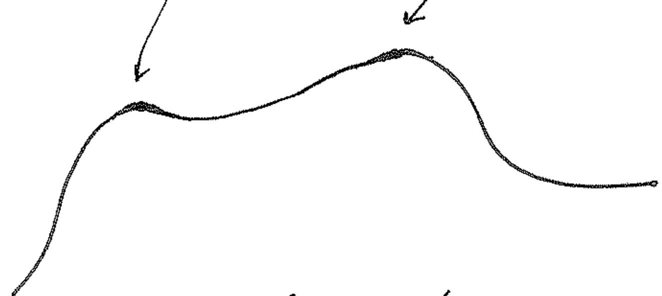
For each x, y, z, we have 4 unknowns.

c_0, c_1, c_2, c_3

→ need 4 equations
- might have <u>interpolating conditions</u> ①
- " <u>continuity conditions</u> at <u>join points</u>.

①





- or conditions that curve pass <u>close</u> to data points.

EACH type of condition defines different curve from same data

10.4 Interpolation

cubic interpolating polynomial

– not necessarily commonly used, but illustrates principles.

(*)

4 control points: p_0, p_1, p_2, p_3

$$p_k = \begin{pmatrix} x_k \\ y_k \\ z_k \end{pmatrix}$$

Seek C, st. $C\underset{\sim}{u}$ passes through $p_0, \dots, p_3$.
i.e. interpolates.

12 unknowns.

$p_0, \dots, p_3$ give 12 equations

Assume $p_0 = p_0(0)$, $p_1 = p_1(\frac{1}{3})$, $p_2 = p_2(\frac{2}{3})$, $p_3 = p_3(1)$
equally spaced.

Conditions:

$$p_0 = p(0) = c_0$$

$$p_1 = p(\tfrac{1}{3}) = c_0 + \tfrac{1}{3}c_1 + (\tfrac{1}{3})^2 c_2 + (\tfrac{1}{3})^3 c_3$$

$$p_2 = p(\tfrac{2}{3}) = c_0 + \tfrac{2}{3}c_1 + (\tfrac{2}{3})^2 c_2 + (\tfrac{2}{3})^3 c_3$$

$$p_3 = p(1) = c_0 + c_1 + c_2 + c_3$$

$$p = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

For each x, y, z,
$\Rightarrow$ $p = Ac$;

Vandermonde matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{3}^2 & \frac{1}{3}^3 \\ 1 & \frac{2}{3} & \frac{2}{3}^2 & \frac{2}{3}^3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad c = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Or, together

$$\begin{pmatrix} p_{0x} & p_{0y} & p_{0z} \\ p_{1x} & p_{1y} & p_{1z} \\ p_{2x} & p_{2y} & p_{2z} \\ p_{3x} & p_{3y} & p_{3z} \end{pmatrix} = A \begin{pmatrix} c_{0x} & c_{0y} & c_{0z} \\ c_{1x} & c_{1y} & c_{1z} \\ c_{2x} & c_{2y} & c_{2z} \\ c_{3x} & c_{3y} & c_{3z} \end{pmatrix}$$

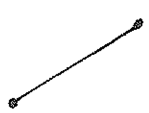
4×3 $\quad$ 4×4 $\quad$ 4×3

$$\mathbb{P} = \mathbb{A} \; \mathbb{C}$$

A is non-singular, and we can invert it

$M_I = A^{-1}$ "interpolating geometry matrix"

$$\boxed{C = M_I P}$$

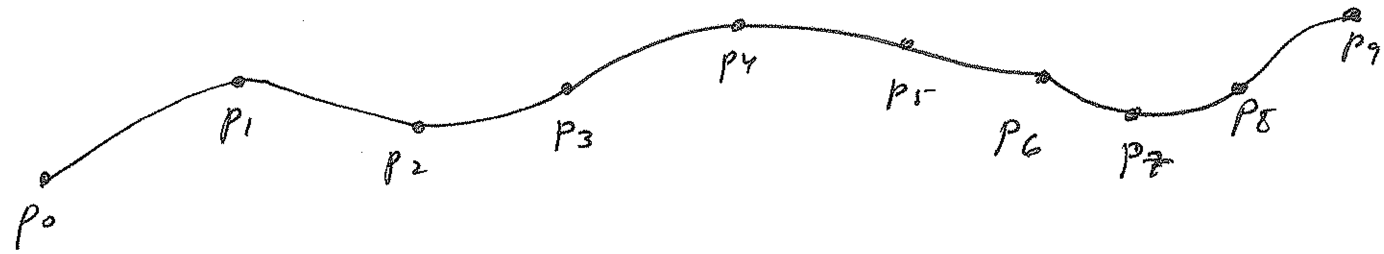


2 points
unique
degree 1
poly. curve

3 points
unique
degree 2.

Rather than degree m polynomial, set of cubic polynomials

E.g. $\circledast$ continue.

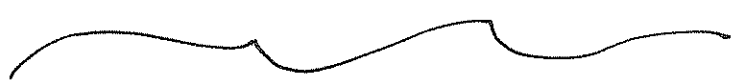


- First use $p_0, \cdots, p_3$ to get one cubic.
- then use $p_3, \cdots, p_6$ to get another cubic

$\Rightarrow$ Continuity at the join point.

Note: A and hence M_I is the same for each segment if we take $u = 0, \frac{1}{3}, \frac{2}{3}, 1$ for each segment.

Problem: derivatives at the join points discont.



$\leftarrow$ may get something like this.

10.4.1 Blending Functions

$$\underset{\sim}{p}(u) = \underset{\sim}{u}^T \underset{\sim}{c} \quad \Rightarrow \quad \begin{pmatrix} 1 & u & u^2 & u^3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\underset{\sim}{p}(u) = \begin{pmatrix} x(u) & y(u) & z(u) \end{pmatrix} = \begin{pmatrix} 1 & u & u^2 & u^3 \end{pmatrix} \underbrace{\begin{pmatrix} c_{0x} & c_{0y} & c_{0z} \\ c_{1x} & c_{1y} & c_{1z} \\ c_{2x} & c_{2y} & c_{2z} \\ c_{3x} & c_{3y} & c_{3z} \end{pmatrix}}_{C}$$

Recall, we determined C as

$C = M_I P$, so substituting

$$p(u) = u^T (M_I P)$$

$$p(u) = \underbrace{(u^T M_I)}_{\text{data-independent}} P$$

P ← depends on data

$$p(u) = b(u)^T P, \quad \text{where } \boxed{b(u) = M_I^T u}$$

blending polynomials

$$b(u) = \begin{bmatrix} b_0(u) \\ b_1(u) \\ b_2(u) \\ b_3(u) \end{bmatrix}$$

← each b.p. is a cubic

$$\boxed{\underset{\sim}{p}(u) = \sum_{i=0}^{3} b_i(u) \underset{\sim}{p}_i}$$

blend together effect of different control points.

Blending functions

Lagrange basis polynomials

$$b_0(u) = -\frac{9}{2}\left(u-\frac{1}{3}\right)\left(u-\frac{2}{3}\right)(u-1)$$

$$b_1(u) = \frac{27}{2}\, u\left(u-\frac{2}{3}\right)(u-1)$$

$$b_2(u) = -\frac{27}{2}\, u\left(u-\frac{1}{3}\right)(u-1)$$

$$b_3(u) = \frac{9}{2}\, u\left(u-\frac{1}{3}\right)\left(u-\frac{2}{3}\right)$$

roots in $[0,1]$

Note :

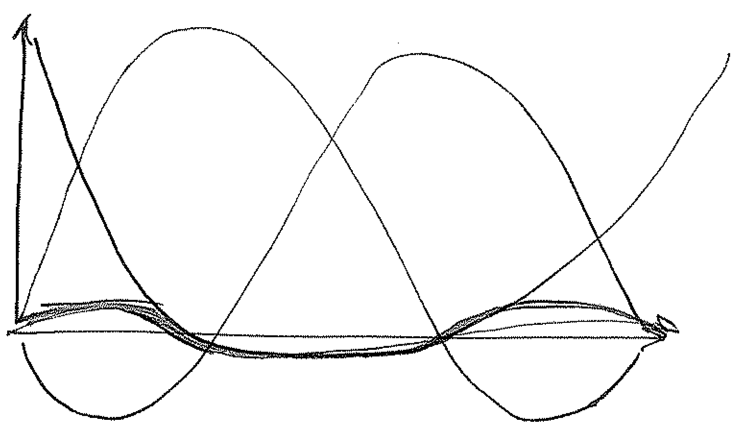
$$b_0(0) = +\frac{9}{2}\left(+\frac{1}{3}\right)\left(+\frac{2}{3}\right)(+1) = \frac{18}{18} = 1 \ \checkmark$$

$$b_1\left(\frac{1}{3}\right) = \frac{27}{2}\left(\frac{1}{3}\right)\left(+\frac{1}{3}\right)\left(+\frac{2}{3}\right) = \frac{27\cdot 2}{2\cdot 27} = 1 \ \checkmark$$

$$b_2\left(\frac{2}{3}\right) = +\frac{27}{2}\ \frac{2}{3}\ \frac{1}{3}\ +\frac{1}{3} = \frac{27\cdot 2}{2\cdot 27} = 1 \ \checkmark$$

$$b_3(1) = \frac{9}{2}\ 1\left(+\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{9\cdot 2}{2\cdot 9} = 1 \ \checkmark$$

Can be derived this way...



blend polynomials for interpolation

Problems :

- already quite oscillatory looking - worse for higher order
- kinks at join points.

10.4.2 Cubic Interpolating Patch.

natural extension of interpolating curve

> example of separable surfaces
> $p(u,v) = f(u)g(v)$
>
> tensor product surface.

Bicubic Surface Patch

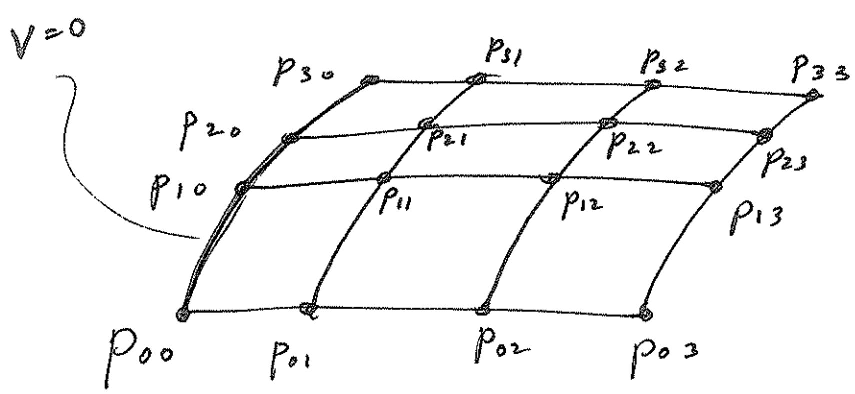
$$p(u,v) = \sum_{i=0}^{3} \sum_{j=0}^{3} u^i v^j c_{ij}$$

$$\mathbf{p}(u,v) = \underset{\sim}{u}^T C \underset{\sim}{v} \quad , \quad u = \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix}, v = \begin{pmatrix} 1 \\ v \\ v^2 \\ v^3 \end{pmatrix}$$

C has $4 \times 4 = \mathbf{16}$ elements for each dimension (x,y,z) for a total of **48** elements

have enough equations for unknowns

16 control points.



take
$u = 0, \frac{1}{3}, \frac{2}{3}, 1$

$v = 0, \frac{1}{3}, \frac{2}{3}, 1$

To solve, first consider $v = 0$.

$$\mathbf{p}(u,0) = \sum_{i=0}^{3} \sum_{j=0}^{3} u^i 0^j c_{ij} = \sum_{i=0}^{3} u^i c_{i0}$$

Curve in u that interpolates $p_{00}, p_{10}, p_{20}, p_{30}$.

$$p(u,0) = (u^T M_I)\, P_{:,0}$$

$$= u^T M_I \begin{bmatrix} p_{00} \\ p_{10} \\ p_{20} \\ p_{30} \end{bmatrix}$$

$$= u^T C \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Similarly for $p(u, \frac{1}{3})$, $p(u, \frac{2}{3})$, $p(u, 1)$

Write all 16 equations as.

$$u^T M_I P = u^T C A^T \qquad A = M_I^{-1}$$

$$\Rightarrow \boxed{C = M_I P M_I^T}$$

$$p(u,v) = u^T (M_I P M_I^T) v$$

$$= u^T M_I \; P \; M_I^T v$$

$$= (M_I^T u)^T P (M_I^T v)$$

$$= b(u)^T \mathbf{P} \; b(v)$$

$$= \sum_{i=0}^{3} \sum_{j=0}^{3} b_i(u)\, \mathbf{p}_{ij}\, b_j(v)$$

$b_i(u)\, b_j(v)$ blending patch.

control points are weighed by blending patches.

Algebraic form:

$$p(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = \underset{\sim}{u}^T c$$

Geometric form:

(interpolating cubic) $$p(u) = F(u) p_0 + F_1(u) p_1 + F_2(u) p_2 + F_3(u) p_3$$

$$p(u) = b(u)^T p$$

coefficients from data

$$c = M_I p$$

~~Fo~~ Interpolation

$$p(0) = c_0$$
$$p(1) = c_0 + c_1 + c_2 + c_3$$
$$p\left(\tfrac{1}{3}\right) = c_0 + \left(\tfrac{1}{3}\right) c_1 + \left(\tfrac{1}{3}\right)^2 c_2 + \left(\tfrac{1}{3}\right)^3 c_3$$
$$p\left(\tfrac{2}{3}\right) = c_0 + \left(\tfrac{2}{3}\right) c_1 + \left(\tfrac{2}{3}\right)^2 c_2 + \left(\tfrac{2}{3}\right)^3 c_3$$

$$\Rightarrow \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \frac{2}{3} & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$p = A c \qquad M_I = A^{-1}$$
$$c = M_I p$$

Algebraic $p(u) = u^T c$ $\quad$ p in terms of coefficients

Geometric $p(u) = b(u)^T p$ $\quad$ p in terms of data

$$Mp = c$$

$$\boxed{p(u) = u^T M p = (u^T M)\, p.}$$

$$\boxed{b(u) = M^T u}$$

$$p(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix}$$

Cubic Interpolating Curves & Surfaces

(treating x, y, z separately).

$$x(u) = c_0 + c_1 u + c_2 u^2 + c_3 u^3 = \begin{pmatrix} 1 & u & u^2 & u^3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$x(u) = u^T c$$

<u>Interpolation</u> of points p_0, p_1, p_2, p_3

$$(0, x_0), \ \left(\tfrac{1}{3}, x_1\right), \ \left(\tfrac{2}{3}, x_2\right), \ (1, x_3)$$

$$\underbrace{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{P} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{3}\right) & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \left(\frac{2}{3}\right) & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_4 \end{pmatrix}}_{c}$$

4 equations & 4 unknowns.

interpolating geometry matrix

Let $M_I = A^{-1}$

A is a Vandermonde matrix

Then

$$c = M_I P \quad \text{coefficients}$$

This approach corresponds to using a monomial basis.

It is also possible to use a Lagrange basis

$$x(u) = b_0(u)x_0 + b_1(u)x_1 + b_2(u)x_2 + b_3(u)x_3$$

where

$$b_0(u) = \frac{(u-\frac{1}{3})(u-\frac{2}{3})(u-1)}{(0-\frac{1}{3})(0-\frac{2}{3})(0-1)} = -\frac{9}{2}(u-\tfrac{1}{3})(u-\tfrac{2}{3})(u-1)$$

$$b_1(u) = \frac{(u-0)(u-\frac{2}{3})(u-1)}{(\frac{1}{3}-0)(\frac{1}{3}-\frac{2}{3})(\frac{1}{3}-1)} = \frac{27}{2}(u)(u-\tfrac{2}{3})(u-1)$$

$$b_2(u) = \frac{(u-0)(u-\frac{1}{3})(u-1)}{(\frac{2}{3}-0)(\frac{2}{3}-\frac{1}{3})(\frac{2}{3}-1)} = -\frac{27}{2}u(u-\tfrac{1}{3})(u-1)$$

$$b_3(u) = \frac{(u-0)(u-\frac{1}{3})(u-\frac{2}{3})}{(1-0)(1-\frac{1}{3})(1-\frac{2}{3})} = \frac{9}{2}u(u-\tfrac{1}{3})(u-\tfrac{2}{3})$$

$$x(u) = u^T c = u^T M_I \underset{\sim}{x}$$

$$= b(u)^T x$$ blending functions (or Lagrange polynomials)

$$x(u) = b_0(u)x_0 + b_1(u)x_1 + b_2(u)x_2 + b_3(u)x_3$$

$$x(u,v) = \sum_{i=0}^{3}\sum_{j=0}^{3} u^i v^j c_{ij} = \underset{\sim}{u}^T C \underset{\sim}{v}$$

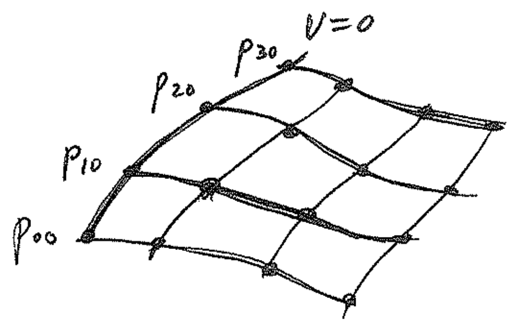
assume our data is given at $u = 0, \frac{1}{3}, \frac{2}{3}, 1$

$v = 0, \frac{1}{3}, \frac{2}{3}, 1$

16 eqs., 16 unknowns

16×16 linear system.

$\Rightarrow$ But each curve is also an interpolating cubic



e.g. $v = 0$

$\Rightarrow$ decouples into 4, 4×4 systems.

$$x(u,0) = u^T M_I \begin{pmatrix} x_{00} \\ x_{10} \\ x_{20} \\ x_{30} \end{pmatrix}$$

$$= \sum_{i=0}^{3}\sum_{j=0}^{3} u^i (0)^j c_{ij} = \sum_{i=0}^{3} u^i c_{i0} = u^T C \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Similarly, $v = \frac{1}{3}$

$$x(u,\tfrac{1}{3}) = \sum_{i=0}^{3}\sum_{j=0}^{3} u^i \left(\tfrac{1}{3}\right)^j c_{ij} = u^T C \begin{pmatrix} 1 \\ \frac{1}{3} \\ (\frac{1}{3})^2 \\ (\frac{1}{3})^3 \end{pmatrix} = u^T C\, v(\tfrac{1}{3})$$

$$u^T M_I X = u^T C A^T$$

$$M_I X = C A^T$$

$$\Rightarrow \quad C = M_I X A^{-T} = M_I X M_I^T$$

$$\boxed{C = M_I X M_I^T}$$

$$x(u,v) = \underset{\sim}{u}^T C \underset{\sim}{v}$$

$$= (u^T M_I) X (M_I^T \underset{\sim}{v})$$

$$= (M_I^T u)^T X (M_I^T v)$$

$$= b(u)^T X \; b(v)$$

$$= \sum_{i,j} b_i(u) \, X_{ij} \, b_j(v)$$

$b_i(u) \, b_j(v)$ Blending patch