Homework 3
CS 210

Question	Points	Score
1	8	
2	6	
3	8	
4	8	
5	14	
6	14	
7	14	
8	14	
9	14	
Total	100	

Linear Systems

1. For each of the following statements, indicate whether the statement is true or false.

   T/F If a matrix A is singular, then the number of solutions to the linear system $A\mathbf{x} = \mathbf{b}$ depends on the particular choice of right-hand-side $\mathbf{b}$.

   T/F If a matrix A is nonsingular, then the number of solutions to the linear system $A\mathbf{x} = \mathbf{b}$ depends on the particular choice of right-hand-side $\mathbf{b}$.

   T/F If a matrix has a very small determinant, then the matrix is nearly singular.

   T/F If any matrix has a zero on its main diagonal, then it is necessarily singular.

2. Can a system of linear equations $A\mathbf{x} = \mathbf{b}$ have exactly two solutions? Explain your answer.

LU Factorization and Gaussian Eliminiation

3. For each of the following statements, indicate whether the statement is true or false.

   T/F If a triangular matrix has a zero on its main diagonal, then it is necessarily singular.

   T/F The product of two upper triangular matrices is upper triangular.

   T/F If a linear system is well-conditioned, then pivoting is unnecessary in Gaussian elimination.

   T/F Once the LU factorization of a matrix has been computed to solve a linear system, then subsequent linear systems with the same matrix but different right-hand-side vectors can be solved without refactoring the matrix.

4. (T&B 20.2) Suppose $A \in \mathbb{R}^{n\times n}$ has an LU factorization. Suppose that A is banded with bandwidth $2p+1$, i.e., $a_{ij} = 0$ for $|i-j| > p$. What can you say about the sparsity patterns of the factors L and U of A? Explain.

5. Consider LU factorization with partial pivoting of the matrix A which computes

   $$M_{n-1}P_{n-1}\cdots M_3P_3M_2P_2M_1P_1A = U$$

   where P_i is a row permutation matrix interchanging rows i and $j > i$.

(a) Show that the matrix $P_3 P_2 M_1 P_2^{-1} P_3^{-1}$ has the same structure as the matrix M_1.

(b) Explain how the above expression is transformed into the form $PA = LU$, where P is a row permutation matrix.

Cholesky Factorization

6. (Heath 2.37) Suppose that the symmetric $(n+1) \times (n+1)$ matrix

$$B = \begin{pmatrix} \alpha & \mathbf{a}^T \\ \mathbf{a} & A \end{pmatrix}$$

is positive definite.

(a) Show that the scalar α must be positive and the $n \times n$ matrix A must be positive definite.

(b) What is the Cholesky factorization of B in terms of α, $\mathbf{a}$, and the Cholesky factorization of A?

Singular Value Decomposition

7. (T&B 4.1) Determine SVDs of the following matrices (by hand calculation):

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

8. Let A be an $m \times n$ singular matrix of rank r with SVD

$$A = U\Sigma V^T = \left(\begin{array}{c|c|c|c} & & & \\ \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \\ & & & \end{array} \right) \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \vdots \\ \hline \mathbf{v}_n^T \end{pmatrix}$$

$$= \begin{pmatrix} \hat{U} & \tilde{U} \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \hat{V}^T \\ \tilde{V}^T \end{pmatrix}$$

where $\sigma_1 \geq \ldots \geq \sigma_r > 0$, $\hat{U}$ consists of the first r columns of U, $\tilde{U}$ consists of the remaining $m-r$ columns of U, $\hat{V}$ consists of the first r columns of V, and $\tilde{V}$ consists of the remaining $n-r$ columns of V. Give bases for the spaces range(A), null(A), range(A^T) and null(A^T) in terms of the components of the SVD of A, and a brief justification.

9. Show that for an $m \times n$ matrix of full column rank n, the matrix $A(A^TA)^{-1}A^T$ is an orthogonal projector onto range(A). Hint: use the SVD of A.