Homework 2
CS 210

Question	Points	Score
1	10	
2	20	
3	20	
4	25	
5	15	
6	10	
Total	100	

Matrix algebra

1. (Heath 2.4a) Show that the following matrix is singular.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

2. (Trefethen&Bau 2.6) If $\mathbf{u}$ and $\mathbf{v}$ are $m-$vectors, the matrix $A = I + \mathbf{u}\mathbf{v}^T$ is known as a rank-one pertubation of the identity. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha\mathbf{u}\mathbf{v}^T$ for some scalar α, and give an expression for α. For what $\mathbf{u}$ and $\mathbf{v}$ is A singular? If it is singular, what is null(A)?

3. (Heath 2.8) Let A and B be any two $n \times n$ matrices.
   (a) Prove that $(AB)^T = B^T A^T$.
   (b) If A and B are both non-singular, prove that $(AB)^{-1} = B^{-1}A^{-1}$.

Vector and matrix norms

4. Let $\mathbf{x} \in \mathbb{R}^n$. Two vector norms, $||\mathbf{x}||_a$ and $||\mathbf{x}||_b$, are *equivalent* if $\exists\, c, d \in \mathbb{R}$ such that

$$c||\mathbf{x}||_b \leq ||\mathbf{x}||_a \leq d||\mathbf{x}||_b.$$

   Matrix norm equivalence is defined analogously to vector norm equivalence, i.e., $||\cdot||_a$ and $||\cdot||_b$ are equivalent if $\exists\, c, d$ s.t. $c||A||_b \leq ||A||_a \leq d||A||_b$.

   (a) Let $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$. For each of the following, verify the inequality and give an example of a non-zero vector or matrix for which the bound is achieved (showing that the bound is tight):
      i. $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_2$
      ii. $||\mathbf{x}||_2 \leq \sqrt{n}||\mathbf{x}||_\infty$
      iii. $||A||_\infty \leq \sqrt{n}||A||_2$
      iv. $||A||_2 \leq \sqrt{n}||A||_\infty$

      This shows that $||\cdot||_\infty$ and $||\cdot||_2$ are equivalent, and that their induced matrix norms are equivalent.

   (b) Prove that the equivalence of two vector norms implies the equivalence of their induced matrix norms.

Sensitivity and conditioning

5. (Heath 2.58) Suppose that the $n \times n$ matrix A is perfectly well-conditioned, i.e., cond(A) = 1. Which of the following matrices would then necessarily share this same property?

   (a) cA, where c is any nonzero scalar

   (b) DA, where D is a nonsingular diagonal matrix

   (c) PA, where P is any permutation matrix

   (d) BA, where B is any nonsingular matrix

   (e) A^{-1}, the inverse of A

   (f) A^T, the transpose of A

6. Under what circumstances does a small residual vector $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ imply that $\mathbf{x}$ is an accurate solution to the linear system $A\mathbf{x} = \mathbf{b}$?