Gaussian Elimination with Pivoting (partial.

$$M_{n-1} \cdots M_3 P_3 M_2 P_2 M_1 P_1 A = U$$

$$M_{n-1}' \cdots M_3' M_2' M_1' \underbrace{P_{n-1} \cdots P_3 P_2 P_1}_{P} A = U$$

$$P A = \underbrace{(M_1')^{-1} (M_2')^{-1} (M_3')^{-1} \cdots (M_{n-1}')^{-1}}_{} U$$

$$P A = L U$$

$$M_4 \left(P_4 M_3 P_4^{-1}\right)\left(P_4 P_3 M_2 P_3^{-1} P_4^{-1}\right)\left(P_4 P_3 P_2 M_1 P_2^{-1} P_3^{-1} P_4^{-1}\right) P_4 P_3 P_2 P_1$$

$$\left(M_4' M_3' M_2' M_1'\right) P_4 P_3 P_2 P_1 A = U$$

$$\boxed{PA = LU}$$

Complete Pivoting

$$M_3 P_3 M_2 P_2 M_1 P_1 A Q_1 Q_2 Q_3 = U$$

$$M_3 \left(P_3 M_2 P_3^{-1}\right)\left(P_3 P_2 M_1 P_2^{-1} P_3^{-1}\right) P_3 P_2 P_1 A Q_1 Q_2 Q_3$$

$$M_3' M_2' M_1' P_3 P_2 P_1 A Q_1 Q_2 Q_3 = U$$

$$L^{-1} P A Q = U$$

$$\boxed{PAQ = LU}$$

$$Ax = b$$

partial pivoting

$PA\,x = Pb$

$LU\,x = Pb$

$L(y) = Pb$ solve by for. subst.

$Ux = y$ solve by back. subst.

complete pivoting

$PAQQ^T x = Pb$

$LU\,Q^T x = Pb$

$L\,y = Pb$ solve by for. subst

$U z = y$

$x = Qz$

$\boxed{\S 2.4.7}$ Complexity of Solving Linear Systems.

LU factorization $\boxed{\frac{2}{3}n^3}$ ← dominant phase as $n \to \infty$

forward + backward solve $\boxed{2n^2}$

$\underline{A^{-1}}$

- n linear systems

1. LU $\boxed{\frac{2}{3}n^3}$
2. n forward + backward solves $\boxed{2n^3}$
3. $\boxed{2n^2}$ $+ \times$ $(A^{-1}b)$

$\Rightarrow$ $\boxed{\text{Rarely compute an explicit inverse}}$ $\Leftarrow$

$\boxed{\S 2.5}$ Special Types of linear systems.

- symmetry $A^T = A$
- pos. def $x^T A x > 0 \quad \forall x \neq 0$
- banded $a_{ij} = 0 \quad |i-j| > \beta \quad \beta = \text{bandwidth}$.
  e.g., tridiagonal, $\beta = 1$.
- sparse

$\boxed{\S 2.5.1}$ SPD systems.

$A = LL^T$ or $A = R^T R$

- stable w/o pivoting
- $\frac{n^3}{3}$ $+ \times$ ($\frac{1}{2}$ of LU) – ($\frac{1}{2}$ work $\frac{1}{2}$ storage)
- Cholesky factorization
- also LDL^T factorization

Cholesky Algorithm

$$LL^T$$

$$B = AC$$

$$b_{ij} = \sum_k a_{ik} c_{kj}$$

$$\begin{pmatrix} | & 0 & 0 & & \\ & | & 0 & & \\ & & | & & \\ l_1 & l_2 & l_3 & \ddots & \\ | & | & | & & l_n \end{pmatrix} \begin{pmatrix} - & l_1^T & - \\ 0 & - l_2^T & - \\ & & \vdots \\ & & l_n^T \end{pmatrix}$$

$$\begin{pmatrix} | & | & & | \\ l_1 & l_2 & \cdots & l_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} - & l_1^T & - \\ - & l_2^T & - \\ & \vdots & \\ - & l_n^T & - \end{pmatrix}$$

$$\begin{pmatrix} x & 0 & 0 & & 0 \\ x & x & 0 & & \\ x & x & x & & \\ x & x & x & \ddots & \\ x & x & x & & \ddots \\ x & x & x & & & x \end{pmatrix} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$A = l_1 l_1^T + l_2 l_2^T + \cdots + l_n l_n^T$$

"outer product Cholesky"

to do in place: replace all l's with a's.

$$
\begin{aligned}
&\text{for } k = 1 \cdots n \\
&\quad l_{kk} = \sqrt{a_{kk}} \\
&\quad \text{for } i = k+1 \cdots n \\
&\qquad l_{ik} = \frac{a_{ik}}{l_{kk}} \\
&\quad \text{end} \\
&\quad \text{for } i = k+1 \cdots n \\
&\qquad \text{for } j = k+1 \cdots n \\
&\qquad\quad a_{ij} = a_{ij} - l_{ik} l_{jk} \\
&\qquad \text{end} \\
&\quad \text{end.} \\
&\text{end.}
\end{aligned}
$$

$$l_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad l_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \quad l_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$l_1 l_1^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}; \quad l_2 l_2^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 1 \end{pmatrix}; \quad l_3 l_3^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$A = l_1 l_1^T + l_2 l_2^T + l_3 l_3^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 8 & 4 \\ 3 & 4 & 19 \end{pmatrix}$$

$$l_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad l_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \qquad l_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$A - l_1 l_1^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 10 \end{pmatrix}$$

$$A - l_1 l_1^T - l_2 l_2^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

SPD Systems

$A = A^T$

$x^T A x > 0 \quad \forall\, x \neq 0$

Cholesky factorization

Can get $U = L^T$, i.e. $A = LL^T$
(note L not generally unit triangular)

- no pivoting needed for stability!
- $\sqrt{\ }$ will be of positive numbers
- only access lower triangular portion of A
- $\frac{n^3}{6}$ ⊗'s, and $\sim\frac{n^3}{6}$ ⊕'s $\rightarrow$ $\frac{n^3}{3}$ ops

$\frac{1}{2}$ work of G.E.

$\frac{1}{2}$ storage of G.E.

2.5.2. Symmetric Indefinite Systems

(May have $x^T A x > 0$ or $x^T A x < 0$)

- pivoting may be needed for stability
- Compute

$$PAP^T = LDL^T$$

$\boxed{\text{use symmetric pivoting to retain symmetry}}$

- but may not exist or be stably computable
- so take D tridiagonal or with 1×1 or 2×2 diagonal blocks.
- work + storage comparable to Cholesky

2.5.3 Banded Systems

$$A_{ij} = 0 \quad \text{for } |i-j| > \beta,$$

bandwidth $= \beta$ (or $2\beta + 1$)

w/o pivoting LU have same bandwidth

w/ " may double bandwidth.