$\boxed{\text{Ex. 2.13}}$ G.E.

$$\begin{aligned} x + 2y + 2z &= 3 \\ 4x + 4y + 2z &= 6 \\ 4x + 6y + 4z &= 10 \end{aligned}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 4 & 6 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix}$$

$$M_1 A = \begin{pmatrix} 1 & & \\ -4 & 1 & \\ -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 4 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -4 & -6 \\ 0 & -2 & -4 \end{pmatrix}$$

$$M_2 M_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 0 & -4 & -6 \\ 0 & -2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -4 & -6 \\ 0 & 0 & -1 \end{pmatrix}$$

$$A = \left[M_1^{-1} M_2^{-1} \right] (M_2 M_1 A) = \left[\begin{pmatrix} 1 & & \\ 4 & 1 & \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & \frac{1}{2} & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 2 & 2 \\ 0 & -4 & -6 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ 4 & 1 & \\ 4 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ & -4 & -6 \\ & & -1 \end{pmatrix}$$

Lecture 20. Gaussian Elimination. $\boxed{\text{LECTURE 3}}$

$$\underbrace{M_{n-1} \cdots M_2 M_1}_{L^{-1}} A = U$$

$$A = LU \qquad L \quad \text{"unit lower triangular"}$$

General Formula for M_k

$$M_k a = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & -m_{k+1} & 1 & \\ & & \vdots & & \ddots \\ & & -m_n & & & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

k^{th} column

$$m_i = \frac{a_i}{a_k}$$

a_k is the "pivot"

- M_k is unit lower triangular
- M_k is of the form $\quad I - \vec{m}_k \vec{e}_k^{\,T}$, $\quad \vec{m}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{pmatrix}$

(*) • M_k^{-1} is really easy to compute:

$$M_k^{-1} = I + \vec{m}_k \vec{e}_k^{\,T} \qquad \text{(check)}$$

(*) • L is really easy to compute

$$L^{-1} = M_{n-1} M_{n-2} \cdots M_2 M_1$$

$$L = (\quad \text{"} \quad)^{-1}$$

$$= M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1}$$

$$M_j^{-1} M_k^{-1} = (I + m_j e_j^T)(I + m_k e_k^T) = I + m_j e_j^T + m_k e_k^T + m_j e_j^T m_k e_k^T$$

but $e_j^T m_k = 0$ when $j < k$.

And all terms in product $m_j e_j^T m_k e_k^T$ will have $j<k$

Therefore

$$L = M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1}$$

$$\boxed{L = I + m_1 e_1^T + m_2 e_2^T + \cdots + m_{n-1} e_{n-1}^T}$$

i.e., $L = \begin{bmatrix} 1 & & & & \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ m_{n1} & m_{2n} & \cdots & m_{n-1\,n} & 1 \end{bmatrix}$

<u>Algorithm</u> <u>Gaussian Elimination</u>
<u>LU Factorization</u>

```
for k = 1, ..., n-1                    [for each column
                                          (except last)]
    if a_kk = 0  stop                  [                ]
    for i = k+1 ... n                  [compute elements of L,
        m_ik = a_ik / a_kk                 L_k]
    end
    for j = k+1 ... n                  [update to find
        for i = k+1 ... n                 elements of U]
            a_ij = a_ij - m_ik a_kj      (upper k x n elements
        end                                untouched
    end                                    because M_k^-1 is
                                           the identity there)
```

In place: write m_{ik} directly into lower part of matrix. replace m_{ik} with a_{ik} above

Operation Count

$$\sum_{k=1}^{n-1}\left[\left(\sum_{i=k+1}^{n} 1\right) + \sum_{j=k+1}^{n}\sum_{i=k+1}^{n} 2\right]$$

$$= \sum_{k=1}^{n-1}\left[n-(k+1)+1 + 2\sum_{j=k+1}^{n} n-(k+1)+1\right]$$

$$= \sum_{k=1}^{n-1}\left[n-k + 2\sum_{j=k+1}^{n} n-k\right]$$

$$= \sum_{k=1}^{n-1}\left[n-k + 2(n-k)\left(n-(k+1)+1\right)\right]$$

$$= \sum_{k=1}^{n-1}\left[(n-k) + 2(n-k)(n-k)\right]$$

$$= \sum_{k=1}^{n-1}(n-k)\left[2(n-k)+1\right]$$

$$m = n-k$$

$$k=1 \Rightarrow m = n-1$$

$$k=n-1 \Rightarrow m = n-n+1 = 1$$

$$= \sum_{m=1}^{n-1} m\left[2m+1\right]$$

$$= \sum_{m=1}^{n-1}\left(2m^2+m\right)$$

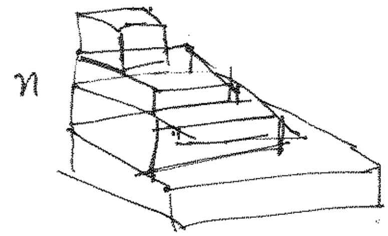
$$= 2\,\frac{(n-1)(n)(2n-2+1)}{6} + \frac{(n-1)n}{2}$$

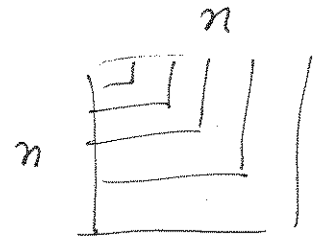
$$= \frac{n(n-1)(2n-1)}{3} + \frac{n(n-1)}{2} \sim \frac{2}{3}n^3$$

$$\boxed{S_n = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

How to estimate operation count geometrically

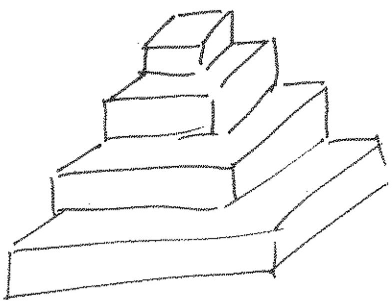
- dominate by operation in inner most loop
  <u>2 flops</u>

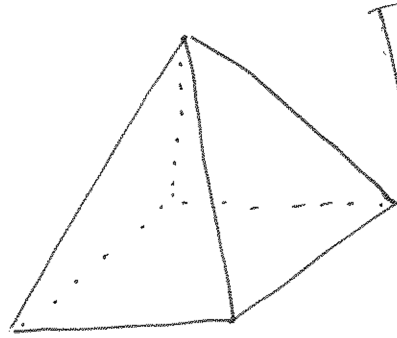




$$V = \frac{1}{3} b h = \frac{1}{3} n^3$$

$$\text{ops} \sim \frac{2}{3} n^3$$





pyramid with
Volume $= \frac{1}{3} b h$

Instability of Gaussian Elimination (w/o pivoting)

Example

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

full rank

$K_2 = (3+\sqrt{5})/2 \approx 2.618$

but G.E. fails right away!

$$A = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix}$$

factors $\quad L = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 10^{-20} & 1 \\ 0 & 1-10^{+20} \end{bmatrix}$

assume after rounding $\quad \tilde{L} = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}, \quad \tilde{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$

$$\tilde{A} = \tilde{L}\tilde{U} = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 0 \end{bmatrix}$$ not close to A!

e.g. $\quad Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x \cong \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\tilde{A}\tilde{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \tilde{x} \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

large error in solve.

Gaussian Elimination (as presented so far) is not stable!

Add pivoting to stabilize.

$$\begin{pmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 4 & 6 & 4 \end{pmatrix}$$

$$\underbrace{\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{P_1} \underbrace{\begin{pmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 4 & 6 & 4 \end{pmatrix}}_{A}}_{\shortparallel}$$

$$\underbrace{\begin{pmatrix} 1 & & \\ -1/4 & 1 & \\ -1 & 0 & 1 \end{pmatrix}}_{M_1} \underbrace{\begin{pmatrix} 4 & 4 & 2 \\ 1 & 2 & 2 \\ 4 & 6 & 4 \end{pmatrix}}_{P_1 A} = \begin{pmatrix} 4 & 4 & 2 \\ 0 & 1 & 3/2 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\underbrace{\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{P_2} \begin{pmatrix} 4 & 4 & 2 \\ 0 & 1 & 3/2 \\ 0 & 2 & 2 \end{pmatrix}}_{\shortparallel}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix}}_{M_2} \begin{pmatrix} 4 & 4 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 3/2 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 4 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & \frac{5}{4} \end{pmatrix}}_{U}$$

$$M_2 P_2 M_1 P_1 A = U$$