2.4 Solving Linear Systems $\quad Ax = b$

transform into a linear system that is easy to solve.
e.g., triangular system:

$$\begin{pmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{pmatrix} \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} = \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix}$$

Q How would you solve it?

If M is non-singular, then solution is not affected by

$$MA\cancel{z} = Mb$$

because

$$\cancel{z} = (MA)^{-1} Mb = A^{-1}M^{-1}Mb = A^{-1}b \checkmark$$
$$= x \checkmark$$

$\boxed{\text{Ex. 2.9}}$ Permutation Matrix

- identity with rows + cols permuted

(row permutation) $\quad PAx = Pb \quad$ reorders the equation
x unchanged

(col permutation)

$$AP\cancel{z} = b \Rightarrow \cancel{z} = (AP)^{-1}b = P^T A^{-1} b$$
$$AP(P^T x) = b \qquad = P^T x \text{ row perm}$$

$\boxed{\text{Ex. 2.10}}$ Diagonal Scaling

$DAx = Db \quad DAz = Db \quad$ row scaling — doesn't change exact soln, but

$ADz = b \quad$ column scaling

$\Rightarrow z = (AD)^{-1}b = D^{-1}A^{-1}b = D^{-1}x \quad$ does change numerical accuracy

$ADD^{-1}z = b$

§ 2.4.2 Triangular Linear Systems.

lower triangle L — $l_{ij} = 0$ when $i < j$

upper triangular U — $u_{ij} = 0$ when $i > j$

$Lx = b$

$\sum_{k=1}^{i} l_{ik} x_k = b_i$

$x_i = \frac{1}{l_{ii}}\left(b_i - \sum_{k=1}^{i-1} l_{ik} x_k\right)$

$Ux = b$

$\sum_{k=i}^{n} u_{ik} x_k = b_i$

$x_i = \frac{1}{u_{ii}}\left(b_i - \sum_{k=i+1}^{n} u_{ik} x_k\right)$

Ex. 2.11 Triangular Linear System 3x3

```
for j = 1 ... n
  if l_jj == 0   stop
  x_j = b_j / l_jj
  for i = j+1 ... n
      b_i = b_i - l_ij x_j
  end
end.
```

§ 2.4.3 Elementary Elimination Matrices

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rightarrow \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1$$ leave first row unchanged

$$\begin{pmatrix} -\frac{y}{x} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \left(-\frac{y}{x} x + y\right) = -y + y = 0.$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{y}{x} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$M_k a = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & -m_{k+1} & 1 & & \\ & & -m_{k+2} & & 1 & \\ & & -m_n & & & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_0 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

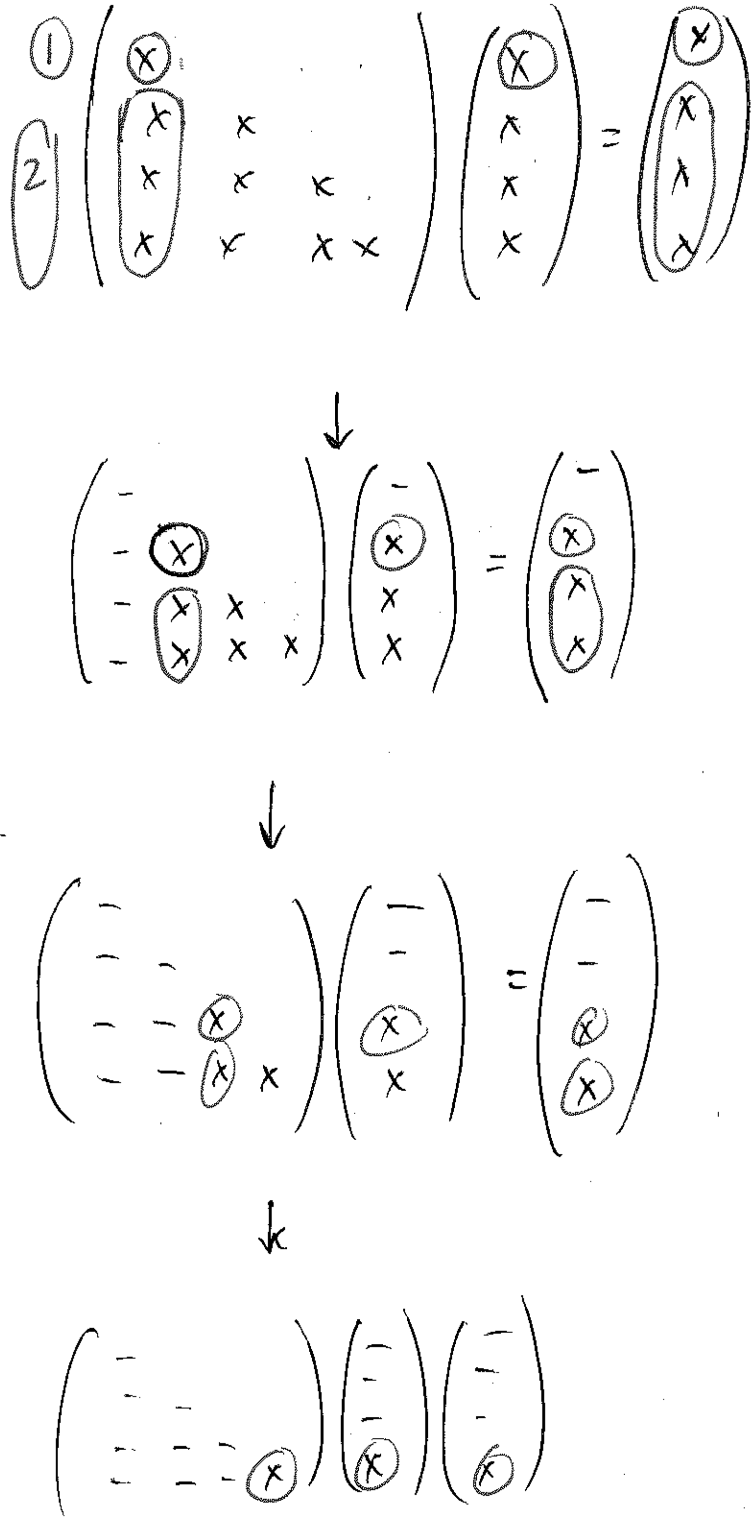
"elementary elimination" matrix or "Gauss Transformation"

$$m_i = \frac{a_i}{a_k}, \quad i = k+1, \dots, n$$

pivot

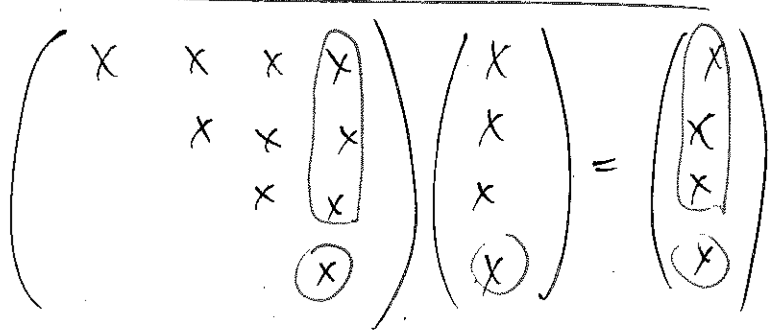
Forward Substitution

for $j = 1 \dots n$
  if $l_{jj} == 0$ stop
① $x_j = b_j / l_{jj}$
② for $i = j+1 \dots n$
    $b_i = b_i - l_{ij} x_j$
  end
end



Backward Substitution

for $j = n \dots 1$
  if $u_{jj} == 0$ then stop
  $x_j = b_j / u_{jj}$
  for $i = j-1 \dots 1$
    $b_i = b_i - u_{ij} x_j$
  end
end



$$\begin{pmatrix} 1 & 0 \\ -\frac{a_2}{a_1} & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ -\frac{a_2}{a_1} \cdot a_1 + a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$$

eliminate element 2 using multiple of element 1

Elementary Elimination Matrix or Gauss Transform

$M_k =$

$$\begin{array}{c|c} & k \\ k & \left(\begin{array}{cccc|ccc} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ \hline & & & -m_{k+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & -m_n & & & 1 \end{array}\right) \end{array} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$m_i = \frac{a_i}{a_k}, \quad i = k+1 \dots n$$

leave first k elements unchanged

pivot

$j > k$

$$\left(\begin{array}{cccc|ccc} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ \hline & & & -m_{j+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & -m_n & & & 1 \end{array}\right) \left(\begin{array}{cc|ccc} 1 & & & & \\ & 1 & & & \\ \hline & -m_{k+1} & 1 & & \\ & \vdots & & \ddots & \\ & -m_n & & & 1 \end{array}\right)$$

$$e_j^T M_k = (0 \cdots 0 1 0 \cdots 0) \begin{pmatrix} 0 \\ \vdots \\ m_{k+1} \\ \vdots \\ m_n \end{pmatrix} = m_j$$

X $M_j M_k = (I - \vec{m}_j \vec{e}_j^T)(I - \vec{m}_k \vec{e}_k^T) = I - m_j e_j^T - m_k e_k^T + \vec{m}_j (e_j^T m_k) e_k^T$

$= I - \vec{m}_j \vec{e}_j^T - \vec{m}_k \vec{e}_k^T + m_j \vec{m}_j \vec{e}_k^T$

✓ $M_k M_j = (I - \vec{m}_k \vec{e}_k^T)(I - \vec{m}_j \vec{e}_j^T) = I - \vec{m}_k \vec{e}_k^T - \vec{m}_j \vec{e}_j^T + \vec{m}_k \cancel{(\vec{e}_k^T \vec{m}_j)}^{0} \vec{e}_j^T$

$= I - \vec{m}_k \vec{e}_k^T - \vec{m}_j \vec{e}_j^T$

✓ $M_k^{-1} M_j^{-1} = (I + m_k e_k^T)(I + m_j e_j^T) = I + m_k e_k^T + m_j e_j^T$

① M_k - lower triangular - unit diagonal - non-singular.

② $M_k = I - \vec{m}_k \vec{e}_k^T$, $\quad m_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1} \\ \vdots \\ m_n \end{pmatrix}$

③ $M_k^{-1} = I + \vec{m}_k \vec{e}_k^T$

check

$$(I - m_k e_k^T)(I + m_k e_k^T)$$
$$= I + \cancel{m_k e_k^T} - \cancel{m_k e_k^T} - m_k \overbrace{(e_k^T m_k)}^{=0} e_k^T$$
$$= I \checkmark$$

④ $j > k$

$$M_j M_k = (I - m_j e_j^T)(I - m_k e_k^T)$$
$$= I - m_j e_j^T - m_k e_k^T + m_j e_j^T m_k e_k^T \quad \times$$
$$M_k M_j = (I - m_k e_k^T)(I - m_j e_j^T)$$
$$= I - m_k e_k^T - m_j e_j^T + m_k \underbrace{e_k^T m_j}_{=0} e_j^T$$
$$= I - m_k e_k^T - m_j e_j^T$$
$$M_k^{-1} M_j^{-1} = (I + m_k e_k^T)(I + m_j e_j^T)$$
$$= I + m_k e_k^T + m_j e_j^T + m_k \overbrace{(e_k^T m_j)}^{0} e_j^T$$

④ and true $M_l^{-1} M_k^{-1} M_j^{-1} = I + m_l e_l^T + m_k e_k^T + m_j e_j^T$ for $l < k, j$

§2.4.4 Gaussian Elimination & LU Factorization

$$M_1 A x = M_1 b \qquad a_{11} \text{ pivot}$$
$$M_2 M_1 A x = M_2 M_1 b$$
$$\vdots$$
$$\underbrace{M_{n-1} \cdots M_2 M_1}_{M} A x = M_{n-1} \cdots M_2 M_1 b$$

this one is easy to write down using ④

$$\underbrace{MA}_{\text{upper triangular}} x = Mb$$

$$M^{-1}(MA) x = b \quad \Rightarrow A = LU$$
$$\underbrace{\boxed{(M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1}) MA}}_{=A} x = b$$
$$L U x = b$$

$$LUx = b \qquad \text{Ⓠ How to solve now?}$$

LU factors can be reused for different right hand sides

4x4 LU Factorization

$$\left(\begin{array}{c|ccc} x & x & x & x \\ \hline x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{array}\right)$$

$$\begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & & 1 & \\ m_{41} & & & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & & & \\ -m_{21} & 1 & & \\ -m_{31} & & 1 & \\ -m_{41} & & & 1 \end{pmatrix} \left(\begin{array}{c|ccc} x & x & x & x \\ \hline x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{array}\right)}$$

$$\begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & & 1 & \\ m_{41} & & & 1 \end{pmatrix} \left(\begin{array}{c|ccc} x & x & x & x \\ \hline 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{array}\right) = M_1^{-1} M_1 A$$

$$M_1^{-1} M_2^{-1} M_2 M_1 A$$

$$\underbrace{\begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & & 1 & \\ m_{41} & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & m_{32} & 1 & \\ & m_{42} & 0 & 1 \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & -m_{32} & 1 & \\ & -m_{42} & & 1 \end{pmatrix} \left(\begin{array}{cc|cc} x & x & x & x \\ 0 & x & x & x \\ \hline 0 & x & x & x \\ 0 & x & x & x \end{array}\right)}$$

$$\begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & m_{32} & 1 & \\ m_{41} & m_{42} & 0 & 1 \end{pmatrix} \qquad \left(\begin{array}{cc|cc} x & x & x & x \\ 0 & x & x & x \\ \hline 0 & 0 & x & x \\ 0 & 0 & x & x \end{array}\right)$$

$$\begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & m_{32} & 1 & \\ m_{41} & m_{42} & m_{43} & 1 \end{pmatrix} \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix} \qquad (M_1^{-1} M_2^{-1} M_3^{-1})(M_3 M_2 M_1 A)$$

$$= L\,U$$