1) Intro (T&B, Lecture 1)

Lecture 2

matrix A vector x vector b Ax mat-vec

entries a_{ij} x_i b_i αx s-v

2) A linear map $\boxed{b = Ax}$

3) Matrix-Vector Multiplication $\cancel{b_i = \sum_{j=1}^{n} a_{ij} x_j}$

$$\begin{bmatrix} \\ \circ \\ \\ \end{bmatrix} = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ & & & & \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$ "row" view of matrix multiplication

$$\begin{bmatrix} x \\ x \\ x \\ x \\ x \end{bmatrix} = \begin{bmatrix} \text{———} \\ \text{———} \\ \text{—o—o—o—o—o—} \\ \text{———} \\ \text{———} \end{bmatrix} \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}$$ "row view" of matrix multiplication

$b \quad = \quad A \quad x$

$$\boxed{b_i = \sum_{k=1}^{n} a_{ik} x_k}$$ "row view"

4)

$$\begin{bmatrix} x \\ x \\ x \\ x \\ x \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ | & | & | & | & | \\ | & | & | & | & | \end{bmatrix} \begin{bmatrix} x \\ x \\ x \\ x \\ x \end{bmatrix}$$ "column view"

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\vec{b} = \sum_{j=1}^{n} x_j \vec{a}_j = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n$$

$\vec{b}$ is linear combination of $\vec{a}_j$.

Ex. Vandermonde Matrix

Matrix - Matrix

$$B = AC$$

$$b_{ij} = \sum_{k=1}^{n} a_{ik} c_{kj}$$

$$\vec{b}_j = \sum_{k=1}^{n} \vec{a}_k \, c_{kj}$$

Ex. Outer product

$$\vec{u}\vec{v}^T$$

$$\begin{bmatrix} | \\ \vec{u} \\ | \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1\vec{u} & v_2\vec{u} & \cdots & v_n\vec{u} \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} v_1 u_1 & v_2 u_1 & & v_n u_1 \\ v_1 u_2 & v_2 u_2 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ v_1 u_n & v_2 u_n & & v_n u_n \end{bmatrix}$$

Ex upper triangular, U

$$B = AU = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

$$b_j = \sum_{k=1}^{j} a_k$$

Range & Nullspace.

range : set of vectors that can be expressed as

"column space"

Ax

i.e. space ~~of~~ spanned by columns of A.

nullspace . vectors x ~~the~~ s.t.

$Ax = \vec{0}$

rank :

dim (col space)

inverse

$A^{-1}A = AA^{-1} = I$.

2.1. Linear systems

linear transformation $\mathbb{R}^n \to \mathbb{R}^n$, $y, x \in \mathbb{R}^n$

$$\cancel{f(x)}$$
$$\cancel{f(\alpha x) = \alpha f(x)}$$
$$\cancel{f(x+y) = f(x) + f(y)}$$

$$f(x) = Ax = y$$

2.2. Existence & Uniqueness

A $n \times n$ matrix

A nonsingular if any one of:

① has an inverse A^{-1}
$$AA^{-1} = A^{-1}A = I$$

② $\det(A) \neq 0$

③ $\operatorname{rank}(A) = n$

④ if $z \neq 0$, $Az \neq 0$. (no nullspace)

A nonsingular $\Rightarrow$ $Ax = b$ has unique solution for any b.

$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$

A singular $\Rightarrow$ # of solutions depends on b.

$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

- $b \in \operatorname{span}(A)$ infinitely many solutions
- $b \notin \operatorname{span}(A)$ no solutions.

Ex. 2.2

② $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \gamma \\ \frac{4-2\gamma}{3} \end{pmatrix}$

$2x + 3y = 4 \Rightarrow y = \frac{4-2x}{3}$

$4x + 6\left(\frac{4-2x}{3}\right) = 4x - \frac{12x}{3} + \frac{24}{3} = 8 = 8$ ✓

① $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$, unique sol'n

③ $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

no sol'n.

§ 2.3 Sensitivity & Conditioning

$$Ax = b$$

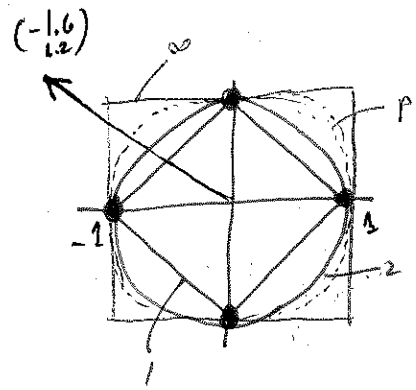
if we <u>perturb</u> the <u>data</u> (A & b), what happens to the <u>solution</u> x?

To <u>measure</u>, "size" of vectors & matrices.

$\boxed{\text{Vector Norms}}$

unit sphere

<u>1-norm</u> $$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$



<u>2-norm</u> $$\|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$$

<u>p-norm</u> $$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$$

<u>∞-norm</u> $$\|x\|_\infty = \max_{1\le i\le n} |x_i|$$

Norm properties

1. $\|x\| > 0$ if $x \neq 0$
2. $\|\alpha x\| = |\alpha| \, \|x\|$
3. $\|x+y\| \le \|x\| + \|y\|$

similar <u>qualitative</u> results. Different norms convenient

- See different <u>unit</u> <u>"spheres"</u>
  - the norm of a vector is the factor by which the corresponding sphere must be expanded or shrunk to encompass the vector

$\boxed{\text{Ex.}}$ $\vec{x} = \begin{pmatrix} -1.6 \\ 1.2 \end{pmatrix}$ $\|x\|_1 = 2.8$, $\|x\|_2 = 2.0$, $\|x\|_\infty = 1.6$

For any x, $\|x\|_1 \ge \|x\|_2 \ge \|x\|_\infty$

Also $\|x\|_1 \le \sqrt{n}\,\|x\|_2$, $\|x\|_2 \le \sqrt{n}\,\|x\|_\infty$, $\|x\|_1 \le n\|x\|_\infty$

§2.3.2 Matrix Norms

induced matrix norm:

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}| \qquad \text{[max column sum]}$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \qquad \text{[max row sum]}$$

(agree w/ corresponding vector norms for $n \times 1$ matrix)

<u>Example</u>

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix} \Rightarrow \|A\|_1 = 6, \quad \|A\|_\infty = 8$$

<u>Matrix norm properties</u>

Δ norm

1. $\|A\| > 0$ if $A \neq 0$
2. $\|\alpha A\| = |\alpha| \, \|A\|$
3. $\|A + B\| \leq \|A\| + \|B\|$

For p-norms

4. $\|AB\| \leq \|A\| \, \|B\|$
5. $\|Ax\| \leq \|A\| \, \|x\|$

"submultiplicative"