CS 210
Midterm

Spring 2017

Name	
Student ID	
Signature	

You may not ask any questions during the exam. If you believe that there is something wrong with a question, write down what you think the question is trying to ask, and answer that.

Question	Points	Score
1	3	
2	3	
3	3	
4	3	
5	3	
6	3	
7	3	
8	3	
9	3	
10	3	
11	4	
12	4	
13	4	
14	4	
15	4	
16	4	
17	4	
18	4	
19	18	
20	20	
Total	100	

True/False

For each question, indicate whether the statement is true or false by circling T or F, respectively.

1. (T/F) Division of two positive floating point numbers may cause overflow.
2. (T/F) The condition number (in 2-norm) of $A^T A$ is the same as the condition number (in 2-norm) of A.
3. (T/F) A good algorithm will produce an accurate solution regardless of the conditioning of the problem being solved.
4. (T/F) If A is nonsingular, then $A\mathbf{x} = \mathbf{b}$ may have more than one solution.
5. (T/F) Gaussian elimination can be used to compute a triangular factorization of a matrix.
6. (T/F) Any symmetric real matrix has a Cholesky factorization.
7. (T/F) The singular value decomposition of a matrix A will give orthonormal bases for range(A), null(A), range(A^T), and null(A^T).
8. (T/F) $\mathbf{x}$ is a solution to the least squares problem $\min_{\mathbf{x}} ||b - A\mathbf{x}||_2$ if and only if $A^T A\mathbf{x} = A^T b$.
9. (T/F) The QR decomposition can be stably computed through the classical Gram-Schmidt algorithm.
10. (T/F) A Householder matrix is an reflection matrix.

Multiple Choice

Instructions: For the multiple choice problems, circle exactly one of (a) - (e).

11. Which one statement about floating point numbers is true?
    (a) If two numbers are exactly representable in floating point, then the result of an arithmetic operation on them is also an exactly representable floating point number.
    (b) Floating point addition is commutative, but not associative.
    (c) Floating point numbers are distributed uniformly througout their range.
    (d) In a unnormalized floating point system, the representation of a number is unique.
    (e) None of the above.

12. Which one of the following statements is false?
    (a) A symmetric matrix, A, satisfies $||A||_1 = ||A||_\infty$.
    (b) A permutation matrix, P, satisfies $||P||_2 = 1$.
    (c) An orthogonal matrix, Q, satisfies $||Q||_2 = 1$.
    (d) If A is singular matrix, then $||A||_2 = 0$.
    (e) For any vector $\mathbf{x}$, $||\mathbf{x}||_1 \geq ||\mathbf{x}||_\infty$.

13. Let A be an $n \times n$ matrix. Which of the following properties would necessarily imply that A is singular?

    I. The columns of A are linearly dependent.

    II. A has a singular value that is 0.

    III. $A\mathbf{z} = \mathbf{0}$, for some $\mathbf{z} \neq \mathbf{0}$.

    (a) II only
    (b) I and II only
    (c) I and III only
    (d) II and III only
    (e) I, II and III

14. Which of the following statements are true?

    I. A problem is ill-conditioned if its solution is highly sensitive to changes in its data.

    II. We can improve conditioning of a problem by switching from single to double precision arithmetic.

    III. In order to solve a problem numerically, it is necessary to have both a well-conditioned problem and a stable algorithm.

    (a) I only
    (b) II only
    (c) I and III only
    (d) II and III only
    (e) I, II and III

15. Which of the following statements are true?

    I. The number of solutions of $A\mathbf{x} = \mathbf{b}$ never depends on $\mathbf{b}$.

    II. If A is singular, then $A\mathbf{x} = \mathbf{b}$ has either no solution or infinitely many solutions.

    III. If $A\mathbf{x} = \mathbf{b}$ then $\mathbf{b}$ must be in the columnspace of A.

    (a) II only
    (b) I and II only
    (c) I and III only
    (d) II and III only
    (e) I, II and III

16. Which of the following statements about the Singular Value Decomposition (SVD) are true?

    I. Every real matrix has an SVD.

    II. If a matrix Q is orthogonal, then its singular values are all 1.

    III. A matrix with rank r will have exactly r singular values that are greater than 0.

    (a) I only
    (b) I and II only
    (c) I and III only
    (d) II and III only
    (e) I, II and III

17. Let $A = U\Sigma V^T$ be the Singular Value Decomposition (SVD) of the matrix A and let A^+ denote the pseudoinverse of A. Which of the following statements are true?

    I. The SVD reveals the rank of a matrix.

    II. $A^+ = U\Sigma^+ V^T$ where Σ^+ is the pseudoinverse of Σ.

    III. The rank of A is the same as the rank of A^+.

    (a) I only
    (b) III only
    (c) I and II only
    (d) I and III only
    (e) I, II and III

18. Which of the following statements about the Least Squares (LS) problem $\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2$ are true?

    I. The solution of the LS problems satisfies $A^T A\mathbf{x} = A^T\mathbf{b}$.

    II. The solution of the LS problem is always unique.

    III. If $\mathbf{b} \in \text{Range}(A)$, then the LS problem has a residual of norm 0.

    (a) I only
    (b) III only
    (c) I and II only
    (d) I and III only
    (e) I, II and III

Written Response

19. *LU Factorization.* Consider the 3×3 matrix

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 6 & 14 & 10 \\ 4 & 10 & 10 \end{pmatrix}.$$

(a) Find unit lower triangular matrices M_1 and M_2 such that $M_2 M_1 A = U$ where U is an upper triangular matrix.

(b) Express A as $A = LU$ where L is a unit lower triangular matrix, and U is the upper triangular matrix you found above.

(c) Explain how you would use the factors L and U to solve the linear equations $A\mathbf{x} = \mathbf{b}$.

20. *Least Squares.* Let $A \in \mathbb{R}^{m \times n}$, where $m > n$. Consider the least squares (LS) problem

$$\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2.$$

(a) Assume A has full rank. Show how you would use the QR decomposition $A = Q\begin{pmatrix} R \\ 0 \end{pmatrix}$ to solve the LS problem.

(b) Now assume A is rank-deficient with rank $r < n$. Show how you would use the Singular Value Decomposition $A = U\Sigma V^T$, with $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$, to solve the LS problem.

(c) In parts (a) and (b) is the solution unique? Why or why not?