True/False

For each question, indicate whether the statement is true or false by circling T or F, respectively.

1. (T/F) If λ is an eigenvalue of A, then it is possible that $|\lambda| > ||A||_2$.
2. (T/F) Inverse iteration on a matrix A will recover the eigenvector associated with the lagest eigenvalue of A.
3. (T/F) The fixed point iteration $x_{k+1} = g(x_k)$ converges near a solution x^* if $|g(x^*)| < 1$. $|g'(x^*)| < 1$
4. (T/F) If a function $f(x)$ is continuous on the interval $x \in [a,b]$, with $f(a) > 0$ and $f(b) < 0$, then f has a fixed point in the interval $[a,b]$.
5. (T/F) Golden Section Search is a linearly convergent algorithm used to find the minimum of a function that is unimodal on an interval $[a,b]$.
6. (T/F) The function $f(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ has a local maximum at $(0,0)$.
7. (T/F) The Conjugate Gradients method for finding the minimum of a quadratic function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + c$, with A symmetric positive definite, converges in at most n iteration (in exact arithmetic).
8. (T/F) The performance of the Conjugate Gradients method for solving $A\mathbf{x} = \mathbf{b}$ can be improved by using a good preconditioner.

① $Ax = \lambda x$

$$||Ax||_2 = ||\lambda x||_2 = |\lambda| \, ||x||_2 = ||Ax||_2 \le ||A||_2 \, ||x||_2$$

$$\Rightarrow \boxed{|\lambda| \le ||A||_2}$$

②

$Ax = \lambda x$

inv. iteration $A^{-1}x = \frac{1}{\lambda} x$

→ smallest λ of A ~> largest of A^{-1}

Multiple Choice

Instructions: For the multiple choice problems, circle exactly one of (a) - (e).

9. Which of the following statements regarding eigenvalue problems are true?

   T I. If $A\mathbf{v} = \mathbf{0}$, then $\mathbf{v}$ is an eigenvector of A.

   F II. If the eigenvalues of a 4×4 matrix A are $-4, 2, 1$, and 3, then the spectral radius of A is $\rho(A) = 3$.

   T III. If $\mathbf{v}$ is an eigenvector of A, then so is $\alpha\mathbf{v}$ for any nonzero scalar α.

   (a) I only
   (b) II only
   (c) III only
   (d) I and II only
   (e) I and III only

$\lim \frac{\|e_{k+1}\|}{\|e_k\|^r} = C$

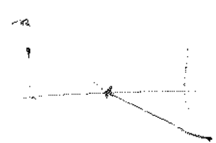
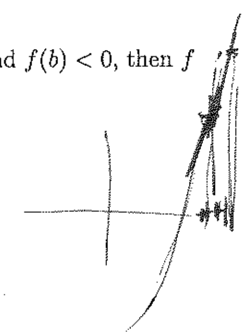
10. Which of the following statements are true?

   T I. If the errors in successive iterations of an algorithm are $10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}$..., then the algorithm is exhibiting quadratic convergence.

   F II. Evaluating a nonlinear function $f(x)$ in a region where $f'(x)$ is large is a well-conditioned problem.

   T III. If a function $f(x)$ is continuous on the interval $x \in [a, b]$, with $f(a) > 0$ and $f(b) < 0$, then f has a root in the interval $[a, b]$.

   (a) I only
   (b) II only
   (c) III only
   (d) I and II only
   (e) I and III only

11. Which of the following statements are true?

   F I. A critical point $\mathbf{x}^*$ of f is a minimizer of f if the Hessian matrix $H_f(\mathbf{x}^*)$ is negative definite.

   T II. A necessary condition for f to have a minimum at $\mathbf{x}^*$ is that $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

   T III. The Steepest Descent method for finding a minimum of a function $f(\mathbf{x})$ uses $-\nabla f(\mathbf{x}_k)$ as the search direction for a 1D line search.

   (a) II only
   (b) III only
   (c) I and II only
   (d) II and III only
   (e) I, II, and III

12. Which of the following statements are true?

F I. The step directions in the Conjugate Gradients methods are chosen to be orthogonal, unlike in Steepest Descent. In particular $\mathbf{s}_k^T \mathbf{s}_j = 0$ for $k \neq j$.

T II. The Conjugate Gradients method can be used to solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ given a symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$.

F III. The search directions used by Conjugate Gradients are actually identical to those used by Steepest Descent.

~~(a) I only~~

(b) II only

(c) I and II only

(d) II and III only

(e) None

13. Which of the following statements are true?

F I. Let $A = M - N$ be a splitting of a matrix A. The iteration $Mx_{k+1} = Nx_k + b$ converges when $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A.

F II. Iterative methods such as Power iteration and Gauss-Seidel iteration typically converge in a finite number of steps.

T III. In the iterative Jacobi method for solving $A\mathbf{x} = \mathbf{b}$, we solve a diagonal linear system in each iteration.

(a) I only

~~(b) II only~~

~~(c) III only~~

(d) I and III only

(e) None

Written Response

14. *Eigenvalue problems.* Let $A \in \mathbb{R}^{n\times n}$ have n eigenvalues and associated eigenvectors satisfying

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i, \quad i = 1, \dots, n$$

(a) For each of the following matrices B, give expressions for the eigenvalues and eigenvectors of B in terms of the eigenvalues and eigenvectors of A.

i. $B = cA$, $c \in \mathbb{R}, c \neq 0$.

ii. $B = A - \sigma I$, $\sigma \in \mathbb{R}$.

iii. $B = XAX^{-1}$, $X \in \mathbb{R}^{n\times n}$ invertible.

(b) Now assume that $A \in \mathbb{R}^{4\times 4}$ is symmetric and that for some orthogonal matrix $V \in \mathbb{R}^{4\times 4}$ with columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$,

$$A = V \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix} V^T$$

i. What are the eigenvalues and associated eigenvectors of A?

ii. Which eigenvalue, eigenvector pair will Power Iteration find?

iii. Which eigenvalue, eigenvector pair will Inverse Power Iteration find?

iv. How does the starting guess $\mathbf{x}_0$ affect which eigenvalue, eigenvector pair Rayleigh Quotient Iteration finds?

(a) (i) $B = cA$

$Av = \lambda v$

$cAv = c\lambda v \quad \Rightarrow$ ~~B has eigenvalues $c\lambda_i$ with associated eigenvecs v_i~~

(ii) $B = A - \sigma I$

$Av = \lambda v \quad \Rightarrow$ ~~B has eigenvals $\lambda - \sigma$ wi~~

$A\cancel{v} - \sigma v = \lambda v - \sigma v$

$\Rightarrow (A - \sigma I)v = (\lambda - \sigma)v$

	eigenvalues	eigenvectors
(i)	$c\lambda_i$	v_i
(ii)	$\lambda_i - \sigma$	v_i
(iii)	λ_i	Xv_i

(iii) $B = XAX^{-1}$

$Av = \lambda v$

$XAv = \lambda Xv$

$XAX^{-1}Xv = \lambda Xv$

$BXv = \lambda Xv$

(b) (i) Let $V = \begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix}$

$(4, v_1), (-3, v_2), (1, v_3), (10, v_4)$

(ii) $(10, v_4)$

(iii) $(1, v_3)$

(iv) x_0 will be used to estimate ~~λ as the~~ for shift σ. $\sigma_0 = \frac{x_0^T A x_0}{x_0^T x_0}$. Then inv. iter on $(A - \sigma_0 I)$ converges to (λ, v) closest to σ_0

15. *Optimization.* Consider the function

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + c,$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric.

(a) Assume A is nonsingular. What are the critical points of $f(\mathbf{x})$? [illegible]

(b) Show that Newton's Method for minimization applied to f converges in one iteration.

(c) Now assume A is singular. Under what conditions does f have one or more critical points?

(a) critical points of f occur where

$$\nabla f(x) = 0$$

$$\nabla f(x)^T \Delta x = \frac{1}{2}\Delta x^T A x + \frac{1}{2} x^T A \Delta x - b^T \Delta x$$

$$= \left(\frac{1}{2}x^T A^T + \frac{1}{2}x^T A\right)\Delta x - b^T \Delta x.$$

$$= \left(\frac{1}{2}x^T(A^T + A) - b^T\right)\Delta x$$

$$\nabla f(x) = \frac{1}{2}(A + A^T)x - b$$

$$= \frac{1}{2}\,2Ax - b$$

$$= Ax - b = 0 \Rightarrow \boxed{Ax = b}$$

$\nabla f(x) = 0$

critical pts of f are those satisfying of $Ax = b$.

(b) Newton's Method:

$$x_{k+1} = x_k - H_f(x_k)^{-1}\nabla f(x_k)$$

$$H_f(x_k)s_k = -\nabla f(x_k)$$

$$\nabla f(x_k) = Ax_k - b \qquad \Rightarrow \qquad As_k = b - Ax_k$$

$$H_f(x_k) = A \qquad \Rightarrow s_k = A^{-1}b - x_k$$

$$x_1 = x_0 + s_0 = x_0 + A^{-1}b - x_0$$

$$= A^{-1}b$$

$\Rightarrow$ x_1 is ~~a~~ unique critical pt.

(c) A singular

$$Ax = b$$

if $b \in \text{Range}(A) \Rightarrow \infty$ many solutions

if $b \notin \text{Range}(A) \rightarrow$ no solutions