Lecture 20. Gaussian Elimination. $\boxed{\text{LECTURE 3}}$ ✓

$$\underbrace{M_{n-1} \cdots M_2 M_1}_{L^{-1}} A = U$$

$$A = LU$$

L "unit lower triangular"

General Formula for M_k

$$M_k a = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & -m_{k+1} & 1 & & \\ & & \vdots & & \ddots & \\ & & -m_n & & & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\uparrow$ k^{th} column

$$m_i = \frac{a_i}{a_k}$$

a_k is the "pivot"

- M_k is unit lower triangular
- M_k is of the form $I - \vec{m}_k \vec{e}_k^{\,T}$, $\vec{m}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{pmatrix}$

(*) • M_k^{-1} is really easy to compute:

$$M_k^{-1} = I + \vec{m}_k \vec{e}_k^{\,T} \quad \text{(check)}$$

(*) • L is really easy to compute

$$L^{-1} = M_{n-1} M_{n-2} \cdots M_2 M_1$$

$$L = (\quad \text{"} \quad)^{-1}$$

$$= M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1}$$

$$M_j^{-1} M_k^{-1} = (I + m_j e_j^T)(I + m_k e_k^T) = I + m_j e_j^T + m_k e_k^T + m_j e_j^T m_k e_k^T$$

but $e_j^T m_k = 0$ when $j < k$.

And all terms in product $m_j e_j^T m_k e_k^T$ will have $j<k$
Therefore

$$L = M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1}$$

$$\boxed{L = I + m_1 e_1^T + m_2 e_2^T + \cdots + m_{n-1} e_{n-1}^T}$$

i.e., $L = \begin{bmatrix} 1 & & & & \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ m_{n1} & m_{2n} & \cdots & m_{n-1 n} & 1 \end{bmatrix}$

Algorithm Gaussian Elimination
LU Factorization

```
for k = 1, ..., n-1
    if a_kk = 0  stop
    for i = k+1 ... n
        m_ik = a_ik / a_kk
    end
    for j = k+1 ... n
        for i = k+1 ... n
(*)         a_ij = a_ij - m_ik a_kj
        end
    end
```

[for each column (except last)]

[compute elements of L, ~~of~~ L_k]

[update to find elements of U]
(upper $k \times$ ~~n~~ elements untouched because M_k^{-1} is the identity there)

In place: write m_{ik} directly into lower part of matrix.
replace m_{ik} with a_{ik} above

(*) $A \cdot B = \sum_{i=1}^{n} \vec{a}_i \vec{b}_i^T$

[| | |] [≡]

Operation Count

$$\sum_{k=1}^{n-1}\left[\left(\sum_{i=k+1}^{n} 1\right) + \sum_{j=k+1}^{n}\sum_{i=k+1}^{n} 2\right]$$

$$= \sum_{k=1}^{n-1}\left[n-(k+1)+1 + 2\sum_{j=k+1}^{n} n-(k+1)+1\right]$$

$$= \sum_{k=1}^{n-1}\left[n-k + 2\sum_{j=k+1}^{n} n-k\right]$$

$$= \sum_{k=1}^{n-1}\left[n-k + 2(n-k)\left(n-(k+1)+1\right)\right]$$

$$= \sum_{k=1}^{n-1}\left[(n-k) + 2(n-k)(n-k)\right]$$

$$= \sum_{k=1}^{n-1} (n-k)\left[2(n-k)+1\right]$$

$$m = n-k$$

$$k=1 \Rightarrow m = n-1$$

$$k=n-1 \Rightarrow m = n-n+1 = 1$$

$$= \sum_{m=1}^{n-1} m\left[2m+1\right]$$

$$\boxed{S_n = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$= \sum_{m=1}^{n-1} \left(2m^2+m\right)$$

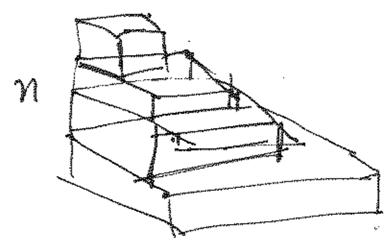
$$= 2\,\frac{(n-1)(n)(2n-2+1)}{6} + \frac{(n-1)n}{2}$$

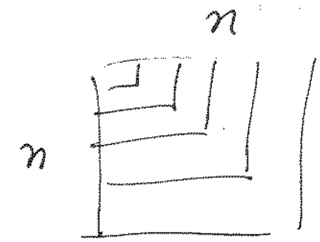
$$= \frac{n(n-1)(2n-1)}{3} + \frac{n(n-1)}{2} \sim \frac{2}{3}n^3$$

dominant term

How to estimate operation count geometrically

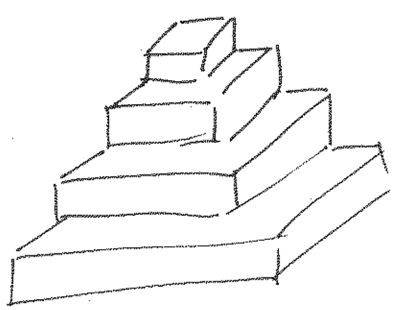
- dominate by operation in inner most loop
  2 flops

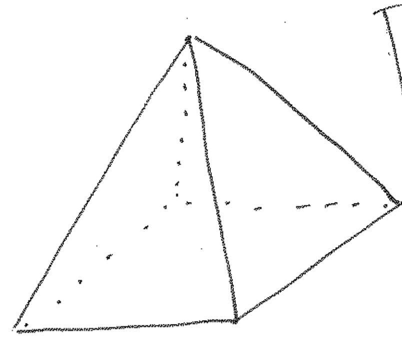




$$V = \frac{1}{3} b h = \frac{1}{3} n^3$$

$$\text{ops} \sim \frac{2}{3} n^3$$





pyramid with
Volume $= \frac{1}{3} b h$

Instability of Gaussian Elimination (w/o pivoting)

Example

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

full rank

$K_2 = (3+\sqrt{5})/2 \approx 2.618$

but G.E. fails right away!

$$A = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix}$$

factors $\quad L = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 10^{-20} & 1 \\ 0 & 1-10^{+20} \end{bmatrix}$

assume after rounding $\quad \tilde{L} = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}, \quad \tilde{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$

$$\tilde{A} = \tilde{L}\tilde{U} = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 0 \end{bmatrix}$$

not close to A! (large backward error)

e.g. $\quad Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x \cong \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\quad \tilde{A}\tilde{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \tilde{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

large error in solve.

Gaussian Elimination (as presented so far) is not stable!

Add pivoting to stabilize.

permute rows so that next pivot is element w/ largest magnitude in column

$\boxed{\text{no pivoting:}}$

$$LU = A$$

$$\begin{pmatrix} 1 & & \\ 4 & 1 & \\ 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ & -4 & -6 \\ & & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 4 & 6 & 4 \end{pmatrix}$$

$\boxed{\text{w/ pivoting}}$

$$\begin{pmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 4 & 6 & 4 \end{pmatrix}$$

$$\underbrace{\underset{P_1}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}\underset{A}{\begin{pmatrix} 1 & 2 & 2 \\ 4 & 4 & 2 \\ 4 & 6 & 4 \end{pmatrix}}}_{=}$$

$$\underset{M_1}{\begin{pmatrix} 1 & & \\ -1/4 & 1 & \\ -1 & 0 & 1 \end{pmatrix}}\underset{P_1 A}{\begin{pmatrix} 4 & 4 & 2 \\ 1 & 2 & 2 \\ 4 & 6 & 4 \end{pmatrix}} = \begin{pmatrix} 4 & 4 & 2 \\ 0 & 1 & 3/2 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\underbrace{\underset{P_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}\begin{pmatrix} 4 & 4 & 2 \\ 0 & 1 & 3/2 \\ 0 & 2 & 2 \end{pmatrix}}_{=}$$

$$\underset{M_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{pmatrix}}\begin{pmatrix} 4 & 4 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 3/2 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 & 4 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & \frac{5}{4} \end{pmatrix}}_{U}$$

$$M_2 P_2 M_1 P_1 A = U$$

$$\underbrace{M_2 P_2 M_1 P_1} A = U$$

$$\underbrace{M_2 P_2 M_1 P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ -\frac{1}{4} & 1 & \\ 1 & & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -\frac{3}{4} & -\frac{1}{2} \end{pmatrix}$$

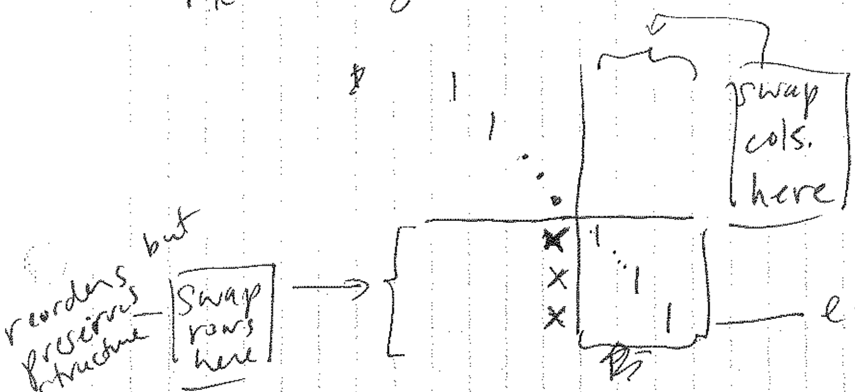
<u>not lower triangular!</u>

$P_k M_j P_k^{-1}$ $\quad j < k \quad$ preserves structure of ~~M~~ $_j$

P_k only modifies rows below jth

P_k^{-1} only modifies cols. after jth

(* can also show this algebraically)



effect here is $PIP^{-1} = I$

So if we can symmetrize above, will preserve structure

$$\underbrace{(M_2)(P_2 M_1 P_2^{-1})}_{L^{-1}} P_2 P_1 A = U$$

$$P_2 P_1 A = LU$$

$$PA = LU$$

$$M_3 P_3 M_2 P_2 M_1 P_1 A$$

$\boxed{4 \times 4}$

$$M_3 (P_3 M_2 P_3^{-1})(P_3 P_2 M_1 P_2^{-1} P_3^{-1})(P_3 P_2 P_1) A = U$$

$$M_3 M_2' M_1' P A = U$$

$$PA = LU$$

$\boxed{5 \times 5}$

$\cancel{M_4}$

$$M_4 P_4 M_3 P_3 M_2 P_2 M_1 P_1 A = U$$

$$M_4 (P_4 M_3 P_4^{-1})(P_4 P_3 M_2 P_3^{-1} P_4^{-1})(P_4 P_3 P_2 M_1 P_2^{-1} P_3^{-1} P_4^{-1})(P_4 P_3 P_2 P_1) A = U$$

$$M_4 M_3' M_2' M_1' P A = U$$

$$PA = LU$$

Note: $P_3 P_2 M_1 P_2^{-1} P_3^{-1}$ has the same <u>form</u> as M_1

$$P_2 M_1 P_2^T = P_2 \left(I - m_1 e_1^T \right) P_2^T$$

$$= I - P_2 m_1 e_1^T P_2^T$$

$$= I - P_2 m_1 e_1^T$$

$$P_3 P_2 M_1 P_2^{-1} P_3^{-1} = P_3 \left(I - P_2 m_1 e_1^T \right) P_3^{-1}$$

$$= I - P_3 P_2 m_1 e_1^T P_3^T$$

$$= \left(I - \underbrace{P_3 P_2 \vec{m}_1}_{\text{permutation of } \vec{m}_1} e_1^T \cdots \right)$$

Gaussian Elimination with Pivoting (partial.

$$M_{n-1} \cdots M_3 P_3 M_2 P_2 M_1 P_1 A = U$$

$$M_{n-1}^{*} \cdots M_3' M_2' M_1' \underbrace{P_{n-1} \cdots P_3 P_2 P_1}_{P''} A = U$$

$$P'' A = \underbrace{(M_1')^{-1} (M_2')^{-1} (M_3')^{-1} \cdots (M_{n-1}')^{-1}}_{L} U$$

$$PA = LU$$

$$M_4 \left(P_4 M_3 P_4^{-1}\right)\left(P_4 P_3 M_2 P_3^{-1} P_4^{-1}\right)\left(P_4 P_3 P_2 M_1 P_2^{-1} P_3^{-1} P_4^{-1}\right) P_4 P_3 P_2 P_1$$

$$\left(M_4^{*} M_3' M_2' M_1'\right) P_4 P_3 P_2 P_1 A = U$$

$$\boxed{PA = LU}$$

Complete Pivoting

$$M_3 P_3 M_2 P_2 M_1 P_1 A Q_1 Q_2 Q_3 = U$$

$$M_3 \left(P_3 M_2 P_3^{-1}\right)\left(P_3 P_2 M_1 P_2^{-1} P_3^{-1}\right) P_3 P_2 P_1 A Q_1 Q_2 Q_3$$

$$M_3' M_2' M_1' P_3 P_2 P_1 A Q_1 Q_2 Q_3 = U$$

$$L^{-1} PAQ = U$$

$$\boxed{PAQ = LU}$$

$$Ax = b$$

partial pivoting	complete pivoting
$PAx = Pb$	$PAQQ^T x = Pb$
$LUx = Pb$	$LU Q^T x = Pb$
$L(y) = Pb$ solve by for. subst.	$Ly = Pb$ solve by for. subst
$Ux = y$ solve by back. subst.	$Uz = y$
	$x = Qz$