Range + Nullspace.

<u>range</u> : set of vectors that can be expressed as

"column space" Ax

i.e. space spanned by columns of A.

<u>nullspace</u> . vectors x s.t.

$$Ax = \vec{0}$$

<u>rank</u> :

dim (col space)

<u>inverse</u>

$$A^{-1}A = AA^{-1} = I.$$

2.1. Linear systems

linear transformation $\mathbb{R}^n \to \mathbb{R}^n$, $y, x \in \mathbb{R}^n$

~~$f(x)$~~

~~$f(\alpha x) = \alpha f(x)$~~

~~$f(x+y) = f(x) + f(y)$~~

$f(x) = Ax = y$

2.2. Existence & Uniqueness

A $n \times n$ matrix

A nonsingular if any one of:

1. has an inverse A^{-1}

$$AA^{-1} = A^{-1}A = I$$

2. $\det(A) \neq 0$
3. $\operatorname{rank}(A) = n$ "full rank"
4. if $z \neq 0$, $Az \neq 0$. (no nullspace)

$\Downarrow$

A nonsingular $\Rightarrow$ $Ax = b$ has unique solution for any b.

$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$

A singular $\Rightarrow$ # of solutions depends on b.

$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

- $b \in \operatorname{span}(A)$ infinitely many solutions
- $b \notin \operatorname{span}(A)$ no solutions.

Ex. 2.2

2. $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \gamma \\ \frac{4 - 2\gamma}{3} \end{pmatrix}$$

$2x + 3y = 4 \Rightarrow y = \frac{4-2x}{3}$

$4x + 6\left(\frac{4-2x}{3}\right) = 4x - \frac{12x}{3} + \frac{24}{3} = 8 = 8$ ✓

3. $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

no sol'n.

1. $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$, unique sol'n

Norms

§2.3 Sensitivity & Conditioning

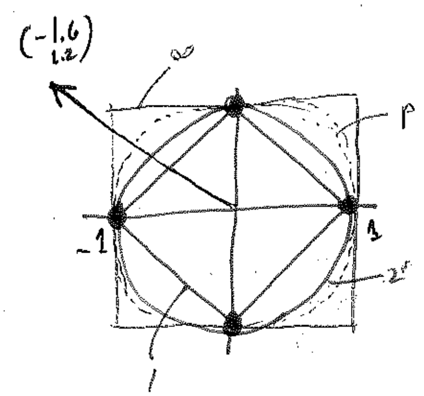
$$Ax = b$$

if we *perturb* the *data* (A & b), what happens to the *solution* x?

To *measure*, "size" of vectors & matrices.

Vector Norms

unit sphere



1-norm $\quad \|x\|_1 = \sum_{i=1}^{n} |x_i|$

2-norm $\quad \|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$

p-norm $\quad \|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$

∞-norm $\quad \|x\|_\infty = \max_{1 \le i \le n} |x_i|$

Norm properties
1. $\|x\| > 0$ if $x \neq 0$
2. $\|\alpha x\| = |\alpha| \, \|x\|$
3. $\|x+y\| \le \|x\| + \|y\|$

similar *qualitative* results. Different norms convenient

- See different *unit* *"spheres"*
  - the norm of a vector is the factor by which the corresponding sphere must be expanded or shrunk to encompass the vector

Ex. $\vec{x} = \begin{pmatrix} -1.6 \\ 1.2 \end{pmatrix} \qquad \|x\|_1 = 2.8, \quad \|x\|_2 = 2.0, \quad \|x\|_\infty = 1.6$

For any x, $\quad \|x\|_1 \ge \|x\|_2 \ge \|x\|_\infty$

Also $\quad \|x\|_1 \le \sqrt{n}\,\|x\|_2, \quad \|x\|_2 \le \sqrt{n}\,\|x\|_\infty, \quad \|x\|_1 \le n\,\|x\|_\infty$

§2.3.2 Matrix Norms

induced matrix norm:

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}| \qquad [\text{max column sum}]$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \qquad [\text{max row sum}]$$

(agree w/ corresponding vector norms for $n \times 1$ matrix)

• Example $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix} \Rightarrow \|A\|_1 = 6, \ \|A\|_\infty = 8$

Matrix norm properties

Δ norm

1. $\|A\| > 0$ if $A \neq 0$
2. $\|\alpha A\| = |\alpha| \, \|A\|$
3. $\|A + B\| \leq \|A\| + \|B\|$

For p-norms

4. $\|AB\| \leq \|A\| \, \|B\|$
5. $\|Ax\| \leq \|A\| \, \|x\|$

"submultiplicative"