Matlab 'eig' $\qquad Av = \lambda Bv$

QZ or generalized Schur decomposition
- ignores symmetry of A & B

Cholesky - symm A + spd B

Algorithm: Rayleigh Quotient Iteration

x_0 = arbitrary non-zero vector
for $k = 1, 2, \ldots$
$\quad \sigma_k = x_{k-1}^T A x_{k-1} / x_{k-1}^T x_{k-1}$ [compute shift]
$\quad$ Solve $(A - \sigma_k I) y_k = x_{k-1}$ [next vector]
$\quad x_k = y_k / \|y_k\|_\infty$ [normalize]
end

- quadratic convergence rate (for non-defective eigenval).
  - cubic for normal matrices (including symm) $AA^T = A^T A$
- but must refactor the matrix each iteration - high cost

4.5.4 Deflation
- Assume found $(\lambda_1, \vec{x}_1)$ s.t. $A\vec{x}_1 = \lambda_1 \vec{x}_1$
- Find H (e.g., Householder) s.t. $H\vec{x}_1 = \alpha \vec{e}_1$

Then $\frac{1}{\alpha}\vec{x}_1 = H^{-1}\vec{e}_1$

$$HAH^{-1}\vec{e}_1 = HA\left(\tfrac{1}{\alpha}\vec{x}_1\right) = \tfrac{1}{\alpha} HA\vec{x}_1 = \frac{\lambda_1}{\alpha} H\vec{x}_1 = \lambda_1 \vec{e}_1$$

i.e. first column of HAH^{-1} is $\lambda_1 \vec{e}_1$

$$HAH^{-1} = \begin{pmatrix} \lambda_1 & \vec{b}^T \\ 0 & \\ \vdots & B \\ 0 & \end{pmatrix}$$

B has eigenvalues $\lambda_2, \ldots, \lambda_n$

- Next, assume

$$B y_2 = \lambda_2 y_2$$

- Then

$$A \begin{pmatrix} \gamma \\ \vec{y}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \vec{b}^T \\ \vec{0} & B \end{pmatrix} \begin{pmatrix} \gamma \\ \vec{y}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 \gamma + \vec{b}^T \vec{y}_2 \\ B \vec{y}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 \gamma + \vec{b}^T \vec{y}_2 \\ \lambda_2 \vec{y}_2 \end{pmatrix}$$

- choose γ s.t.

$$\lambda_1 \gamma + b^T y_2 = \lambda_2 \gamma$$

$$\Rightarrow \quad \gamma = \frac{b^T y_2}{(\lambda_2 - \lambda_1)}$$

- Then $A \begin{pmatrix} \gamma \\ \vec{y}_2 \end{pmatrix} = \lambda_2 \begin{pmatrix} \gamma \\ \vec{y}_2 \end{pmatrix}$

Another approach to deflation (next page) ...

- this general approach (deflation) may lose accuracy
- explicit deflation not recommended if want <u>many</u> $(\lambda, \vec{v})$ pairs
- can enhance accuracy by using inverse iteration with estimated eigenvalue to <u>refine</u>.

Alternative approach to deflation

$$A\vec{x}_1 = \lambda_1 \vec{x}_1$$

Let $\vec{u}_1$ be any vector s.t.

$$\vec{u}_1^T \vec{x}_1 = \lambda_1$$

Then matrix

$$(A - x_1 u_1^T)$$

has eigenvalues $0, \lambda_2, \dots, \lambda_n$

$k=1$

$$(A - x_1 u_1^T) x_1 = A x_1 - x_1 u_1^T x_1 = \lambda_1 x_1 - x_1 \lambda_1 = 0$$

$k = 2, \dots, n$

$$(A - x_1 u_1^T) x_k = A x_k - x_1 u_1^T x_k = \lambda_k x_k - (u_1^T x_k) x_1$$

$$= \lambda_k \left(x_k - \left(\frac{u_1^T x_k}{\lambda_k}\right) x_1 \right)$$

So $\left(x_k - \left(\frac{u_1^T x_k}{\lambda_k}\right) x_1 \right)$ is an ~~eigenvalue~~ eigenvector

for eigenvalue λ_k.

(Check:

$$(A - x_1 u_1^T)\left(x_k - \left(\frac{u_1^T x_k}{\lambda_k}\right) x_1 \right)$$

$$= (A - x_1 u_1^T) x_k - \cancelto{0}{\left(\frac{u_1^T x_k}{\lambda_k}\right)(A - x_1 u_1^T) x_1}$$

$$= \lambda_k \left(x_k - \left(\frac{u_1^T x_k}{\lambda_k}\right) x_1 \right) \checkmark$$

Choices of u_1:

- $u_1 = \lambda_1 x_1$, if A symm. and $\|x_1\|_2 = 1$
- $u_1 = \lambda_1 y_1$ where $A^T y_1 = \lambda_1 y_1$ & $y_1^T x_1 = 1$
- $u_1 = A^T e_k$, if $\|x_1\|_\infty = 1$ + k^{th} component of $(x_1)_k = 1$

Section 4.2.5 : Localizing Eigenvalues

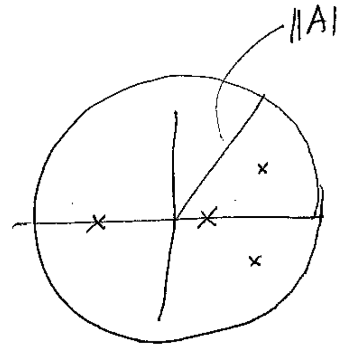
$\boxed{1}$ – can get some crude information about eigenvalues locations

Know $\quad \|\lambda x\| = \|Ax\|$

for vector-induced norm,

$$\Rightarrow \quad |\lambda| \, \|\cancel{x}\| \le \|A\| \cdot \|\cancel{x}\|$$

$$\Rightarrow \quad |\lambda| \le \|A\|$$



$\boxed{2}$ Gershgorin's Theorem : A $n \times n$ matrix

eigenvalues lie in union of n discs, k^{th} disc centered at a_{kk} and radius $\sum_{j \ne k} |a_{kj}|$

Let $\quad A\vec{x} = \lambda \vec{x}$, with $\|\vec{x}\|_\infty = 1$

and $\quad |x_k| = 1 \quad$ (k^{th} component of $\vec{x}$)

$$A\vec{x} = \lambda \vec{x}$$

k^{th} row: $\quad \sum_{j=1}^{n} a_{kj} x_j = \lambda x_k \quad \Rightarrow \quad (\lambda - a_{kk}) x_k = \sum_{j \ne k} a_{kj} x_j$

taking absolute values

$$|\lambda - a_{kk}| \, |x_k| \le \sum_{j \ne k} |a_{kj}| \cdot |x_j|$$

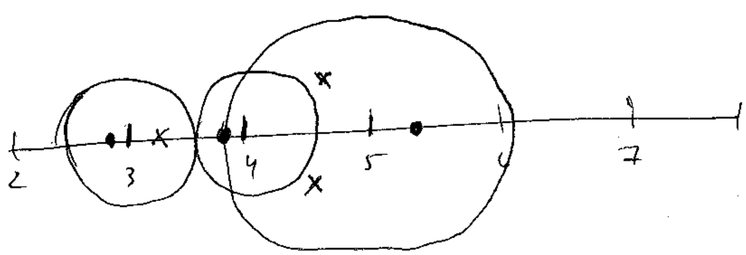
$$\Rightarrow \quad |\lambda - a_{kk}| \le \sum_{j \ne k} |a_{kj}|$$

$$A_2 = \begin{pmatrix} 4 & \frac{1}{2} & 0 \\ \frac{3}{5} & 5 & \frac{3}{5} \\ 0 & \frac{1}{2} & 3 \end{pmatrix}$$

eigenvalues •

Example

$$A_1 = \begin{pmatrix} 4 & -\frac{1}{2} & 0 \\ \frac{3}{5} & 5 & -\frac{3}{5} \\ 0 & \frac{1}{2} & 3 \end{pmatrix}$$



eigenvalues x