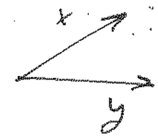
Orthogonality (T&B Lec. 2)

$\boxed{\text{LECTURE 5}}$

Inner Product

$$x^T y = \sum_i x_i y_i = \|x\| \|y\| \cos\theta$$



$$\|x\|_2^2 = x^T x$$

Orthogonal Vectors

x, y orthogonal $\quad x^T y = 0$

orthogonal set $\{x_i\}$ $\quad x_i$ are pairwise orthogonal

- orth. set is lin. indep.

orthonormal set $\quad \|x_i\|_2 = 1$

Components of a vector

$\{q_1, \dots, q_n\}$ orthonormal set

v vector, arbitrary

$(q_i^T v) q_i$ piece of v along q_i

$$v = r + \sum_i (q_i^T v) q_i, \quad r \perp q_i \ \forall i$$

$\{q_i\}$ basis, $r = 0$.

$$v = \sum_i (q_i^T v) q_i = \sum_i q_i q_i^T v$$

Unitary or orthogonal matrix

$$Q Q^T = Q^T Q = I$$

$Q(Q^T b) = b$

interpretation

$$(Qx)^T (Qy) = x^T y$$

Orthogonal Matrix

$Q \in \mathbb{R}^{n \times n}$ is orthogonal if $\left(Q \in \mathbb{C}^{n \times n}, \underline{\text{unitary}} \right.$

$Q^T Q = I$ $\left. Q^* Q = I \right)$

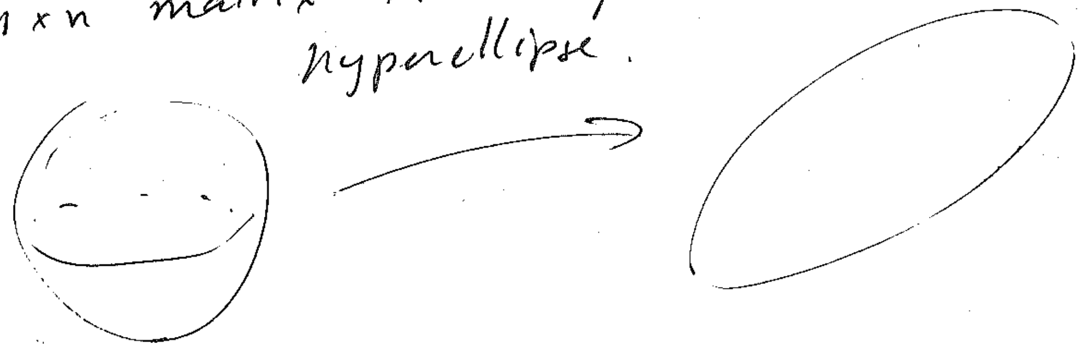
$$\begin{bmatrix} — & \vec{q}_1^{\,T} & — \\ — & \vec{q}_2^{\,T} & — \\ & \vdots & \\ — & \vec{q}_n^{\,T} & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

mult by. Q: rotation ($\det = 1$), or
reflection ($\det = -1$)

SVD

T&B: "Many problems of linear algebra can be better understood if we first ask the question: what if we take the SVD?"

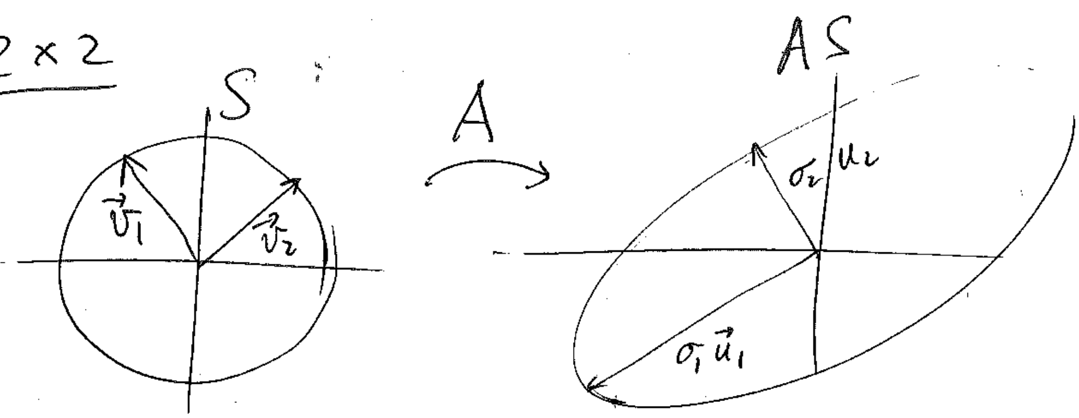
$n \times n$ matrix A maps unit sphere to hyperellipse.



hyperellipse: stretch sphere by factors $\sigma_1, \dots, \sigma_n$ in some orthogonal directions $\vec{u}_1, \dots, \vec{u}_n$

$\sigma_i \vec{u}_i$: principal semiaxes of the hyperellipse. lengths $\sigma_1, \dots, \sigma_n$

2 × 2

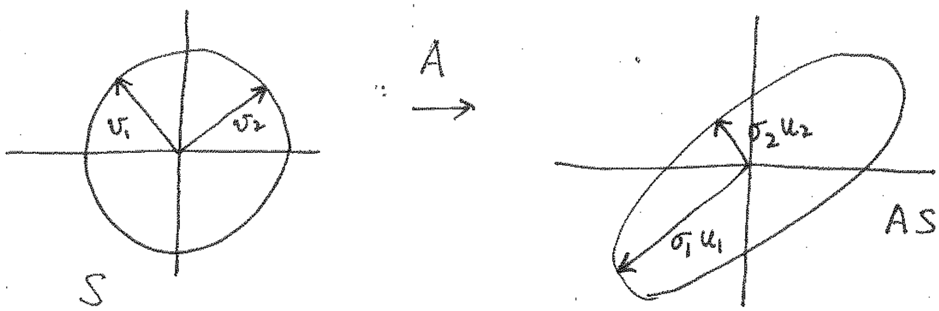


Singular Values: the lengths $\sigma_1, \dots, \sigma_n$ of the n principal semiaxes of ellipse. AS

Left Singular Vectors: $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ in directions of principal semiaxes of AS

Right Singular Vectors: $\vec{v}_1, \dots, \vec{v}_n$ preimages of $\sigma_i \vec{u}_i$ $\quad A\vec{v}_i = \sigma_i \vec{u}_i$

Singular Value Decomposition (T&B Sec. 4)



SVD of a 2×2 matrix

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\Sigma V^T$$

$A \in \mathbb{R}^{m\times n}$, $m \geq n$

A full rank n

- singular values : $\sigma_1, \sigma_2, \ldots, \sigma_n$ — lengths of n principal semi-axes

  convention $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$

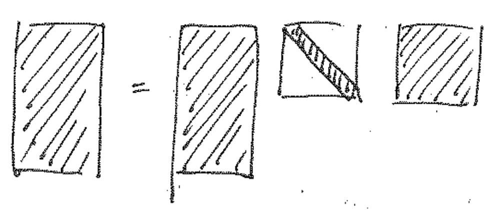
- left singular vectors : $u_1, u_2, \ldots, u_n$ — directions of n principal semi-axes
- right singular vectors : $v_1, v_2, \ldots, v_n$ — preimages of "

$$A v_j = \sigma_j u_j \qquad 1 \leq j \leq n$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}$$

$$A \underset{n\times n}{V} = \underset{m\times n}{\hat{U}}\,\underset{n\times n}{\hat{\Sigma}}$$

$$A = \hat{U}\hat{\Sigma}V^T \qquad \text{Reduced SVD}$$

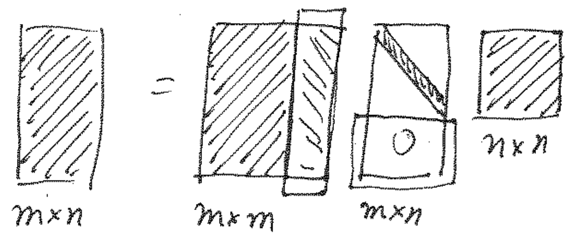


Full SVD

add $m-n$ columns to $\hat{U}$ to get U

complete the basis for $\mathbb{R}^m$

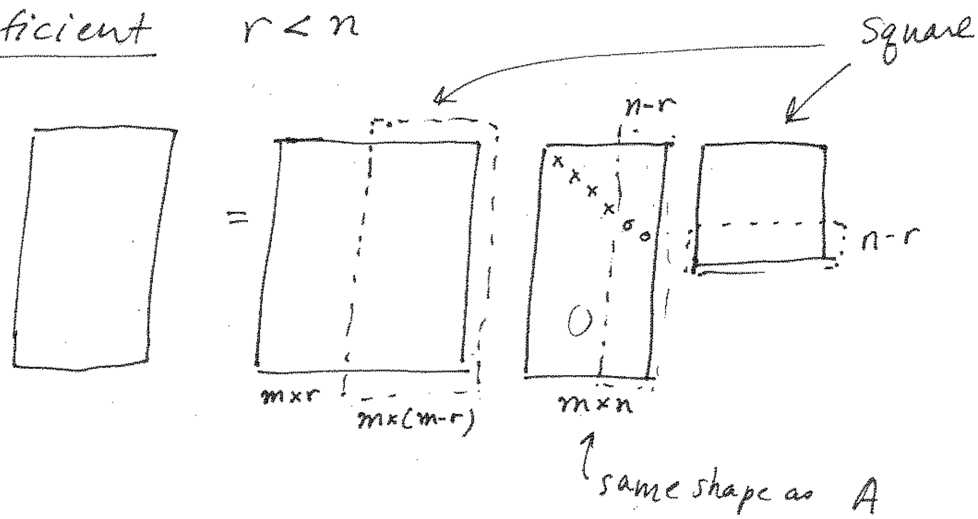
$$A = U \Sigma V^T$$



U unitary $\quad UU^T = U^TU = I$

V unitary $\quad VV^T = V^TV = I$

rank-deficient $\quad r < n$



SVD

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$$

V, U unitary, $\quad \Sigma$ diagonal. $\qquad p = \min(m,n)$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$

Theorem Every $A \in \mathbb{C}^{m\times n}$ has an SVD.

$\{\sigma_i\}$ uniquely determined

(A square + σ_i distinct, u_i, v_i unique up to sign)