Homework 6 (Extra Credit)
CS 210

Question	Points	Score
1	10	
2	10	
3	10	
4	20	
5	20	
6	30	
Total	100	

1. (Heath 11.25)

   (a) What is meant by a *splitting* of a matrix A?

   (b) What form of iterative method for solving a linear system $A\mathbf{x} = \mathbf{b}$ results from such a splitting?

   (c) What condition on the splitting guarantees that the resulting iterative scheme is locally convergent?

   (d) For the matrix

   $$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

   what is the splitting for the Jacobi method?

   (e) For the same matrix as in part (d), what is the splitting for the Gauss-Seidel method?

2. (Heath 11.29) Listed below are several properties that may pertain to various methods for solving systems of linear equation. For each of the properties listed, state whether this quality more accurately describes direct or iterative methods.

   (a) The entries of the matrix are not altered during the computation.

   (b) A prior estimate for the solution is helpful.

   (c) The matrix entries are stored explicitly, using a standard storage scheme such as an array.

   (d) The work required depends on the conditioning fo the problem.

   (e) Once a given system has been solved, another system with the same matrix but a different right-hand side is easily solve.

   (f) Acceleration parameters or preconditioners are usually employed.

   (g) The maximum possible accuracy is relatively easy to obtain.

   (h) “Black box” software is realtively easy to implement.

   (i) The matrix can be defined implicitly by its action on an arbitrary vector.

   (j) A factorization of the matrix is usually performed.

   (k) The amount of work required can often be determined in advance.

3. (Heath 11.6) Prove that the Jacobi iterative method for solving the linear system $A\mathbf{x} = \mathbf{b}$ converges if the matrix A is diagonally dominant by rows. (*Hint:* Use the ∞-norm.)

4. (Heath 11.8) Prove that the successive A-orthogonal search directions generate by the conjugate gradient method satisfy a three-term recurrence, so that each new gradient need be orthogonalized only against the previous two.

5. (Heath 11.9) Show that the subspace spanned by the first m search directions in the conjugate gradient method is the same as the Krylov subspace generated by the sequence $\mathbf{r}_0, A\mathbf{r}_0, A^2\mathbf{r}_0, \ldots, A^{m-1}\mathbf{r}_0$.

6. (Heath 11.12) Implement the steepest descent and conjugate gradient methods for solving symmetric positive definite linear systems. Compare their performance, both in rate of convergence and in total time required, in solving a representative sample of test problems both well-conditioned and ill-conditioned. How does the rate of convergence of the conjugate gradient method compare with the theoretical estimate given in Section 11.5.6? Attach your code.