Homework 4
CS 210

Question	Points	Score
1	6	
2	6	
3	8	
4	6	
5	8	
6	12	
7	6	
8	5	
9	8	
10	12	
11	8	
12	15	
Total	100	

Eigenvalue problems

1. Consider the following statements about eigenvalue problems. Mark each statement as true or false.

   T / F A defective eigenvalue is one where the geometric multiplicity is greater the algebraic multiplicity.

   T / F A good way to compute eigenvalues is by finding roots of the associated characteristic polynomial.

   T / F An orthogonal projection matrix has one eigenvalue equal to 0 and the other eigenvalues equal to 1.

   T / F Symmetric matrices have orthogonal set of eigenvectors.

   T / F A projection matrix must have at least one eigenvalue equal to 0.

   T / F A matrix that has an orthogonal set of eigenvectors can be decomposed as $A = U\Lambda U^T$ where U is orthogonal and Λ is diagonal.

2. (Heath 4.26) Which of the following conditions necessarily imply that an $n \times n$ real matrix A is diagonalizable (i.e., similar to a diagonal matrix)?

   (a) A has n distinct eigenvalues.

   (b) A has only real eigenvalues.

   (c) A is nonsingular.

   (d) A is equal to its transpose.

   (e) A commutes with its transpose.

3. (Heath 4.42)
   (a) If a matrix A has a simple dominant eigenvalue λ_1, what quantity determines the convergence rate of the power method for computing λ_1?
   (b) How can the convergence rate of power iteration be improved?
4. (Heath 4.3) Given an approximate eigenvector $\mathbf{x}$ of A, what is the best estimate (in the least squares sense) for the corresponding eigenvalue?
5. (Heath 4.1)
   (a) Prove that 5 is an eigenvalue of the matrix
$$A = \begin{pmatrix} 6 & 3 & 3 & 1 \\ 0 & 7 & 4 & 5 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$
   (b) Exhibit an eigenvector of A corresponding to the eigenvalue 5.
6. (Heath 4.24) Let A be an $n \times n$ real matrix of rank one.
   (a) Show that $A = \mathbf{u}\mathbf{v}^T$ for some nonzero real vectors $\mathbf{u}$ and $\mathbf{v}$.
   (b) Show that $\mathbf{u}^T\mathbf{v}$ is an eigenvalue of A.
   (c) If power iteration is applied to A, how many iterations are required for it to converge exactly to the eigenvector corresponding to the dominant eigenvalue?

Nonlinear Equations

7. Consider the following statements about nonlinear equation solving. Mark each statement as true or false.

T / F A small residual $||\mathbf{f}(\mathbf{x})||$ guarantees an accurate solution of a system of nonlinear equations $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.

T / F Newton's method is an example of a fixed point iteration.

T / F If an iterative method for solving a nonlinear equation gains more than one bit of accuracy per iteration, then it is said to have a superlinear convergence rate.

T / F Newton's method always converges quadratically.

T / F The nonlinear root-finding problem $f(x) = 0$ has either zero, one, or infinitely many solutions.

T / F A *fixed point* of a function $f(x)$ is a point x^* such that $f(x^*) = 0$.

8. Compare Newton's method and the Secant Method for solving a scalar nolinear equation. What are the advantages and disadvantages of each?
9. (Heath 5.1) Consider the nonlinear equation
$$f(x) = x^2 - 2 = 0.$$
   (a) With $x_0 = 1$, as a starting point, what is the value of x_1 if you use Newton's method for solving this problem?
   (b) With $x_0 = 1$ and $x_1 = 2$ as a starting points, what is the value of x_2 if you use the secant method for the same problem?

10. (Heath 5.12) Newton's method for solving a scalar nonlinear equation $f(x) = 0$ requires computation of the derivative of f at each iteration. Suppose that we instead replace the true derivative with a constant value d, that is, we use the iteration scheme

$$x_{k+1} = x_k - \frac{f(x)}{d}.$$

    (a) Under what condition on the value of d will this scheme be locally convergent?
    (b) What will be the convergence rate, in general?
    (c) Is there any value of d that would still yield quadratic convergence?

11. (Heath 5.10) Carry out one iteration of Newton's method applied ot the system of nonlinear equations

$$\begin{aligned} x_1^2 - x_2^2 &= 0 \\ 2x_1x_2 &= 1 \end{aligned}$$

    with starting value $\mathbf{x}_0 = (0, 1)^T$.

12. *Computer problem* (Heath 5.3) Implement the bisection, Newton, and secant methods for soving nonlinear equations in one dimension, and test your implementation by finding at least one root for each of the following equations. What termination criterion should you use? What convergence rate is achieved in each case?

    (a) $x^3 - 2x - 5 = 0$.
    (b) $e^{-x} = x$.
    (c) $x \sin(x) = 1$.
    (d) $x^3 - 3x^2 + 3x - 1 = 0$.