

Homework 3

CS 210

Question	Points	Score
1	10	
2	15	
3	10	
4	5	
5	10	
6	5	
7	5	
8	5	
9	5	
10	10	
Total	80	

Singular Value Decomposition

1. (T&B 4.1) Determine SVDs of the following matrices (by hand calculation):

(a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

2. Let A be an $m \times n$ singular matrix of rank r with SVD

$$\begin{aligned}
 A = U\Sigma V^T &= \left(\begin{array}{c|c|c|c} & & & \\ \hline \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \\ \hline \end{array} \right) \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \hline \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \vdots \\ \hline \mathbf{v}_n^T \end{pmatrix} \\
 &= \left(\hat{U} \quad \tilde{U} \right) \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \hat{V}^T \\ \tilde{V}^T \end{pmatrix}
 \end{aligned}$$

where $\sigma_1 \geq \dots \geq \sigma_r > 0$, \hat{U} consists of the first r columns of U , \tilde{U} consists of the remaining $m - r$ columns of U , \hat{V} consists of the first r columns of V , and \tilde{V} consists of the remaining $n - r$ columns of V . Give bases for the spaces $\text{range}(A)$, $\text{null}(A)$, $\text{range}(A^T)$ and $\text{null}(A^T)$ in terms of the components of the SVD of A , and a brief justification.

3. Use the SVD of A to show that for an $m \times n$ matrix of full column rank n , the matrix $A(A^T A)^{-1} A^T$ is an orthogonal projector onto $\text{range}(A)$.

Least Squares

4. Consider the least squares problem $\min_{\mathbf{x}} \|\mathbf{b} - A\mathbf{x}\|_2$. Which of the following statements are necessarily true?
- If \mathbf{x} is a solution to the least squares problem, then $A\mathbf{x} = \mathbf{b}$.
 - If \mathbf{x} is a solution to the least squares problem, then the residual vector $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ is in the nullspace of A^T .
 - The solution is unique.
 - A solution may not exist.
 - None of the above.
5. (Heath 3.3) Set up the linear least squares system $A\mathbf{x} \approx \mathbf{b}$ for fitting the model function $f(t, \mathbf{x}) = x_1 t + x_2 e^t$ to the three data points $(1, 2), (2, 3), (3, 5)$. Is the least squares solution unique? Why or why not?
6. (Heath 3.5) Let \mathbf{x} be the solution to the linear least squares problem $A\mathbf{x} \approx \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

Let $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ be the corresponding residual vector. Which of the following three vectors is a possible value for \mathbf{r} ? Why?

- (a) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

Orthogonal and Householder Matrices

7. (Heath 3.23) Which of the following matrices are orthogonal?

- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$
- $\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$

8. (Heath 3.24) Which of the following properties does an orthogonal $n \times n$ matrix necessarily have? (Circle all that apply.)
- (a) It is nonsingular.
 - (b) It preserves the Euclidean vector norm when multiplied times a vector.
 - (c) Its transpose is its inverse.
 - (d) Its columns are orthonormal.
 - (e) It is symmetric.
 - (f) It is diagonal.
 - (g) Its Euclidean matrix norm is 1.
 - (h) Its Euclidean condition number is 1.
9. A Householder matrix H
- (a) has condition number 1.
 - (b) has the property $\|H\|_2 = 1$.
 - (c) is uniquely defined by the condition $H\mathbf{x} = (\alpha, 0, \dots, 0)^T$ where $|\alpha| = \|\mathbf{x}\|_2$.
 - (d) Both (a) and (b).
 - (e) All of the above.
10. Show that a $n \times n$ Householder matrix $H = I - 2\mathbf{v}\mathbf{v}^T/\mathbf{v}^T\mathbf{v}$ has an eigenvalue of 1 with multiplicity $n - 1$ and an eigenvalue of -1 with multiplicity 1.