Homework 2
CS 210

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

Matrix algebra

1. (Trefethen&Bau 2.6) If $\mathbf{u}$ and $\mathbf{v}$ are $m-$vectors, the matrix $A = I + \mathbf{u}\mathbf{v}^T$ is known as a rank-one pertubation of the identity. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha\mathbf{u}\mathbf{v}^T$ for some scalar α, and give an expression for α. For what $\mathbf{u}$ and $\mathbf{v}$ is A singular? If it is singular, what is null(A)?

2. (Heath 2.8) Let A and B be any two $n \times n$ matrices.
   (a) Prove that $(AB)^T = B^T A^T$.
   (b) If A and B are both non-singular, prove that $(AB)^{-1} = B^{-1}A^{-1}$.

Vector and matrix norms

3. Let $\mathbf{x} \in \mathbb{R}^n$. Two vector norms, $||\mathbf{x}||_a$ and $||\mathbf{x}||_b$, are *equivalent* if $\exists\, c, d \in \mathbb{R}$ such that

$$c||\mathbf{x}||_b \leq ||\mathbf{x}||_a \leq d||\mathbf{x}||_b.$$

   Matrix norm equivalence is defined analogously to vector norm equivalence, i.e., $||\cdot||_a$ and $||\cdot||_b$ are equivalent if $\exists\, c, d$ s.t. $c||A||_b \leq ||A||_a \leq d||A||_b$.

   (a) Let $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$. For each of the following, verify the inequality and give an example of a non-zero vector or matrix for which the bound is achieved (showing that the bound is tight):
      i. $||\mathbf{x}||_\infty \leq ||\mathbf{x}||_2$
      ii. $||\mathbf{x}||_2 \leq \sqrt{n}||\mathbf{x}||_\infty$
      iii. $||A||_\infty \leq \sqrt{n}||A||_2$
      iv. $||A||_2 \leq \sqrt{n}||A||_\infty$

      This shows that $||\cdot||_\infty$ and $||\cdot||_2$ are equivalent, and that their induced matrix norms are equivalent.

   (b) Prove that the equivalence of two vector norms implies the equivalence of their induced matrix norms.

Sensitivity and conditioning

4. (Heath 2.58) Suppose that the $n \times n$ matrix A is perfectly well-conditioned, i.e., cond(A) = 1. Which of the following matrices would then necessarily share this same property?

   (a) cA, where c is any nonzero scalar

   (b) DA, where D is a nonsingular diagonal matrix

   (c) PA, where P is any permutation matrix

   (d) BA, where B is any nonsingular matrix

   (e) A^{-1}, the inverse of A

   (f) A^T, the transpose of A

Linear Systems

5. (Heath 2.4a) Show that the following matrix is singular.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

6. For each of the following statements, indicate whether the statement is true or false.

   T/F If a matrix A is singular, then the number of solutions to the linear system $A\mathbf{x} = \mathbf{b}$ depends on the particular choice of right-hand-side $\mathbf{b}$.

   T/F If a matrix A is nonsingular, then the number of solutions to the linear system $A\mathbf{x} = \mathbf{b}$ depends on the particular choice of right-hand-side $\mathbf{b}$.

   T/F If a matrix has a very small determinant, then the matrix is nearly singular.

   T/F If any matrix has a zero on its main diagonal, then it is necessarily singular.

7. Can a system of linear equations $A\mathbf{x} = \mathbf{b}$ have exactly two solutions? Explain your answer.

LU Factorization and Gaussian Eliminiation

8. For each of the following statements, indicate whether the statement is true or false.

   T/F If a triangular matrix has a zero on its main diagonal, then it is necessarily singular.

   T/F The product of two upper triangular matrices is upper triangular.

   T/F If a linear system is well-conditioned, then pivoting is unnecessary in Gaussian elimination.

   T/F Once the LU factorization of a matrix has been computed to solve a linear system, then subsequent linear systems with the same matrix but different right-hand-side vectors can be solved without refactoring the matrix.

9. Consider LU factorization with partial pivoting of the matrix A which computes

$$M_{n-1}P_{n-1} \cdots M_3P_3M_2P_2M_1P_1A = U$$

   where P_i is a row permutation matrix interchanging rows i and $j > i$.

   (a) Show that the matrix $P_3P_2M_1P_2^{-1}P_3^{-1}$ has the same structure as the matrix M_1.

   (b) Explain how the above expression is transformed into the form $PA = LU$, where P is a row permutation matrix.

Cholesky Factorization

10. (Heath 2.37) Suppose that the symmetric $(n+1) \times (n+1)$ matrix

$$B = \begin{pmatrix} \alpha & \mathbf{a}^T \\ \mathbf{a} & A \end{pmatrix}$$

is positive definite.

(a) Show that the scalar α must be positive and the $n \times n$ matrix A must be positive definite.

(b) What is the Cholesky factorization of B in terms of α, $\mathbf{a}$, and the Cholesky factorization of $A - \frac{1}{\alpha}\mathbf{a}\mathbf{a}^T$?