True/False

For each question, indicate whether the statement is true or false by circling T or F, respectively.

1. (T/F) A system of nonlinear equations has either no solutions, exactly one solution, or infinitely many solutions.
2. (T/F) The convergence rate of Newton's Method for solving $f(x) = 0$ depends on f.
3. (T/F) If the errors in successive iterations of an algorithm are $10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, \ldots$, then the algorithm is exhibiting quadratic convergence.

For questions 4-5, consider fixed point iteration for finding a point x^* such that $g(x^*) = x^*$.

4. (T/F) The iteration is locally convergent if $|g'(x^*)| < 1$.
5. (T/F) Newton's Method for solving $f(x) = 0$ is an example of fixed point iteration, with $g(x) = x - f(x)/f'(x)$.

Multiple Choice

Instructions: For the multiple choice problems, circle exactly one of (a) - (e).

6. Which of the following statements about Newton's method for finding a root of a nonlinear equation are true?

   F (a) The cost per iteration of the Secant method is greater than that of Newton's method.

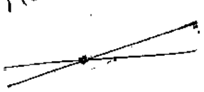
   F (b) Newton's methods exhibits quadratic convergence for any initial guess $\mathbf{x}_0$.

   T (c) Newton's method is an example of a fixed point iteration scheme.

   F (d) When Newton's method converges, then it converges with a quadratic convergence rate.

   (e) None of the above.

$f(x) = 0$



7. Which of the following statements re root finder are true?

   F I. Finding the root of a function which is nearly "flat" around the root is a well-conditioned problem.

   T II. The bisection method has linear convergence with constant 1/2.

   F III. If the errors in successive iterations of an algorithm are $10^{-2}, 10^{-4}, 10^{-6}, \ldots$, then the algorithm is exhibiting quadratic convergence.

   (a) I only

   (b) II only

   (c) III only

   (d) II and III only

   (e) None

8. Which of the following statements are true?

F I. An eigenvector corresponding to a given eigenvalue is unique. identity

T II. Scaling a matrix by a ~~constant~~ scalar c will scale its eigenvalues by that ~~constant~~ scalar.

T III. If a matrix has an eigenvalue of 0, then it is not invertible.

$Ax = \lambda x$

$(cA)x = (c\lambda)x$

(a) I only

(b) II only

(c) ~~III~~ only

(d) II and III only

(e) I, II and III

Written Response

21. *Nonlinear Equations: Newton's Method.* Consider the system of equations

$$x^2 - y^2 = 0$$
$$2xy = 1$$

Carry out one iteration of Newton's Method for finding a solution to this system, with starting value $\mathbf{x}_0 = (0, 1)^T$.

$$f_1(x,y) = x^2 - y^2 = 0$$
$$f_2(x,y) = 2xy - 1 = 0$$

$$\Rightarrow \quad \frac{\partial f_1}{\partial x} = 2x, \quad \frac{\partial f_1}{\partial y} = -2y, \quad \frac{\partial f_2}{\partial x} = 2y, \quad \frac{\partial f_2}{\partial y} = 2x$$

Newton's method

$$\vec{x}_{k+1} = \vec{x}_k + \vec{s}_k$$

$$\cancel{J_f(x_k) s_k = - f(x_k)}$$

$$J_f(\vec{x}_k)\, s_k = -f(\vec{x}_k)$$

$$J_f(x,y) = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{(x,y)} = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$$\vec{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$J_f(\vec{x}_0) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = -f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = -\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} -2s_2 &= 1 \\ 2s_1 &= 1 \end{aligned} \quad \Rightarrow \quad s_1 = \frac{1}{2}, \quad s_2 = -\frac{1}{2}$$

22. *Optimization.* Consider the function

$$\phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + c,$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric.

(a) What are the critical points of ϕ?

(b) How would you classify the critical points of ϕ as maxima, minima or saddle points?

(a) points x, where $\vec{\nabla}\phi(x) = \vec{0}$

$$\nabla\phi(x) = \frac{1}{2}(A + A^T)x - \vec{b}$$

$$= Ax - b, \quad \text{since } A \text{ is symmetric}$$

Setting $\nabla\phi(x) = 0 \implies$

$$Ax - b = 0$$

$$\implies Ax = b$$

$\therefore$ critical pts. of ϕ are those x satisfying

$$Ax = b.$$

(b) by properties of A

if all $\lambda(A) > 0 \implies$ minimum.

all $\lambda(A) < 0 \implies$ max.

some $\lambda(A) < 0$ and some $\lambda(A) > 0$
saddle

some $\lambda(A) = 0$

21. (cont).

$$\begin{pmatrix} x \\ y \end{pmatrix}_1 = \begin{pmatrix} x \\ y \end{pmatrix}_0 + \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}_0$$

$$= \begin{pmatrix} x \\ y \end{pmatrix}_0 + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$\vec{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

General quadratic is

$$f(\vec{x}) = \frac{1}{2}\vec{x}^T A \vec{x} - \vec{b}^T\vec{x} + c$$

$$\nabla f(\vec{x}) = \frac{1}{2}(A^T + A)\vec{x} - b$$

assume A is symmetric, then

$$\nabla f(\vec{x}) = A\vec{x} - b$$

$$H_f(\vec{x}) = A$$

Critical points, x^* such that

$$\nabla f(\vec{x}^*) = 0 \iff A\vec{x}^* = \vec{b}$$

classify critical points by $H_f(x^*) = A$

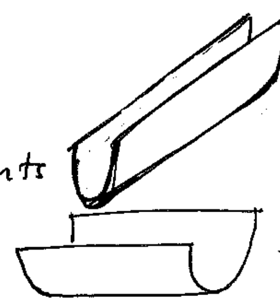
- A positive definite $\Rightarrow$ minimum at x^*
- A negative definite $\Rightarrow$ maximum at x^*
- A has both positive & negative λ, but is invertible (no $\lambda = 0$) $\Rightarrow$ saddle at x^*
- A has one or more zero eigenvalues

  A is singular

  $Ax = b$

  solutions none or $\rightarrow$ no critical points

  infinite $\rightarrow$ infinitely many



e.g. $f(x,y) = x^2 + y$

e.g. $f(x,y) = x^2$