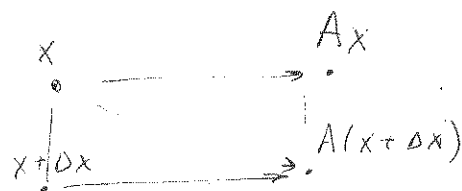


conditioning of Mat-Vec Mult Ax

given x , compute Ax :

perturb input $x + \Delta x$



$$\max_{\Delta x} \frac{\|A(x + \Delta x) - Ax\|}{\|x + \Delta x - x\|} \cdot \frac{\|x\|}{\|Ax\|} =$$

$$= \max_{\Delta x} \frac{\|A\Delta x\|}{\|\Delta x\|} \cdot \frac{\|x\|}{\|Ax\|}$$

$$\leq \|A\| \cdot \frac{\|x\|}{\|Ax\|} = \textcircled{*}$$

and using $\frac{\|x\|}{\|Ax\|} \leq \|A^{-1}\|$

$$\textcircled{*} \leq \|A\| \cdot \|A^{-1}\|$$

Conditioning of $Ax = b$

given b , find x s.t. $Ax = b$. \Leftrightarrow

equivalent to find $x = A^{-1}b$ (for A nonsingular)

$$\therefore \text{cond} \leq \|A^{-1}\| \cdot \|A\|$$

$$\triangleq \text{cond}(A) = \|A\| \cdot \|A^{-1}\|$$

A singular \Rightarrow (convention) $\Rightarrow \text{cond}(A) = \infty$

$$\|A^{-1}\| = \max_{y \neq 0} \frac{\|A^{-1}y\|}{\|y\|}$$

Let $Ax = y, A^{-1}y = x$

$$\|A^{-1}\| = \max_{x \neq 0} \frac{\|x\|}{\|Ax\|}$$

$$= \left(\min_{x \neq 0} \frac{\|Ax\|}{\|x\|} \right)^{-1}$$



Intuition re cond(A)

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

Maximum stretch

$$\|A^{-1}\| = \max_{y \neq 0} \frac{\|A^{-1}y\|}{\|y\|} = \max_{x \neq 0} \frac{\|x\|}{\|Ax\|}, \text{ where } Ax=y$$

$$= \left(\min_{x \neq 0} \frac{\|Ax\|}{\|x\|} \right)^{-1}$$

Minimum stretch

So

$$\text{cond}(A) = \|A\| \cdot \|A^{-1}\| = \max_{x \neq 0}$$

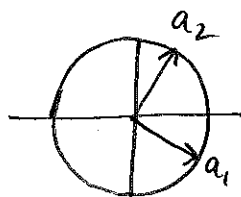
$$= \frac{\max_{x \neq 0} \|Ax\| / \|x\|}{\min_{x \neq 0} \|Ax\| / \|x\|} = \frac{\text{maximum stretch}}{\text{minimum stretch}}$$

How much ~~A~~ mult. by A distorts unit sphere
 A maps unit sphere to a hyperellipse

Example 2.6

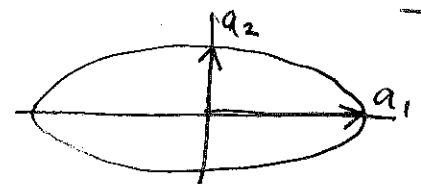
$$A = \begin{bmatrix} 0.87 & 0.5 \\ -0.5 & .87 \end{bmatrix}$$

$$\text{cond}(A) = 1$$



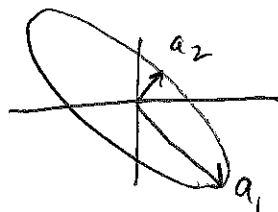
$$A = \begin{bmatrix} 2 & 0 \\ 0 & .5 \end{bmatrix}$$

$$\text{cond}(A) = 4$$



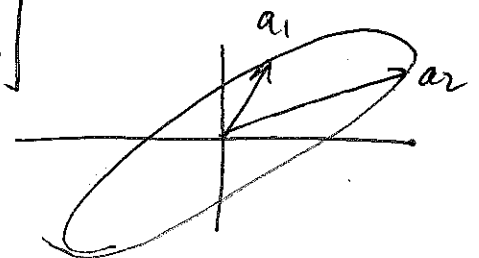
$$A = \begin{bmatrix} 1.73 & 0.25 \\ -1 & 0.43 \end{bmatrix}$$

$$\text{cond}(A) = 4$$



$$A = \begin{bmatrix} 1.52 & 0.91 \\ 0.47 & 0.94 \end{bmatrix}$$

$$\text{cond}(A) = 4$$



Ex. 2.5

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} .5 & 1.5 & -.5 \\ -.5 & 2.5 & .5 \\ -.5 & -.5 & .5 \end{bmatrix}$$

$$\|A\|_1 = 6$$

$$\|A^{-1}\|_1 = 4.5$$

$$\|A\|_\infty = 8$$

$$\|A^{-1}\|_\infty = 3.5$$

$$\text{cond}_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1 = 6 \cdot 4.5 = 27$$

$$\text{cond}_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty = 8 \cdot 3.5 = 28$$

Note: - generally not practical to compute $\text{cond}(A)$ from def'n like this example

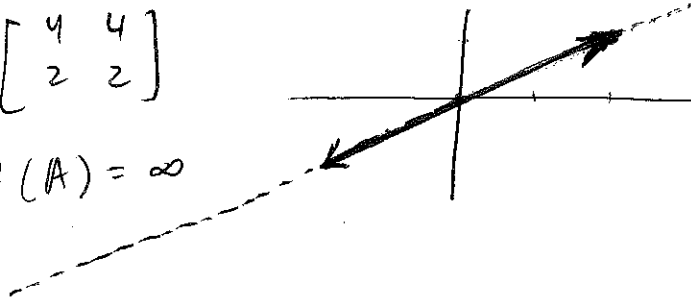
- usually get good estimate of $\text{cond}(A)$ as by product of solution process.

Intuition re $\text{cond}(A)$

How close to singular?

$$A = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$$

$$\text{cond}(A) = \infty$$



but $\det(A)$ not indicative of $\text{cond}(A)$

E.g.

$$\text{cond}(\alpha I) = 1$$

but

$$\det(\alpha I) = \alpha^n$$

Cont.
Ex.
2.6

Properties

1. $\text{cond}(A) \geq 1$ (max stretch/min stretch)
2. $\text{cond}(I) = 1$
3. $\text{cond}(\alpha A) = \text{cond}(A)$
4. D diag, $\text{cond}(D) = \frac{\max_i |d_i|}{\min_i |d_i|}$

§2.3.4. Error Bounds

homework

$\hat{x} = x + \Delta x$ [perturb b] what can we say about $\frac{\|\Delta x\|}{\|x\|}$?

$$A \hat{x} = A(x + \Delta x) = Ax + A\Delta x = b + \Delta b$$

$$\Rightarrow A\Delta x = \Delta b \Rightarrow \boxed{\Delta x = A^{-1}\Delta b}$$

$$\textcircled{1} \|b\| = \|Ax\| \leq \|A\| \|x\| \Rightarrow \|x\| \geq \|b\| / \|A\|$$

$$\textcircled{2} \|\Delta x\| = \|A^{-1}\Delta b\| \leq \|A^{-1}\| \|\Delta b\| \Rightarrow \|\Delta x\| \leq \|A^{-1}\| \|\Delta b\|$$

$$\textcircled{1}, \textcircled{2} \Rightarrow \frac{\|\Delta x\|}{\|x\|} \leq \frac{\|A^{-1}\| \|\Delta b\|}{\|b\| / \|A\|} = \kappa(A) \frac{\|\Delta b\|}{\|b\|}$$

$$\boxed{\frac{\|\Delta x\|}{\|x\|} \leq \kappa \frac{\|\Delta b\|}{\|b\|}}$$

[perturb A]

κ is amplification factor

$$(A + \Delta A)(x + \Delta x) = b$$

$$Ax + (\Delta A)x + \Delta A\Delta x + A\Delta x = b$$

$$\Delta A(x + \Delta x) + A\Delta x = 0$$

$$A^{-1}\Delta A(x + \Delta x) + \Delta x = 0$$

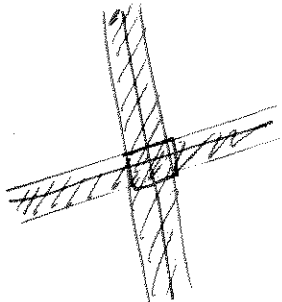
$$\|\Delta x\| \leq \frac{\|A^{-1}\| \|\Delta A\|}{\|\hat{x}\|}$$

$$\frac{\|\Delta x\|}{\|\hat{x}\|} \leq \|A^{-1}\| \|\Delta A\|$$

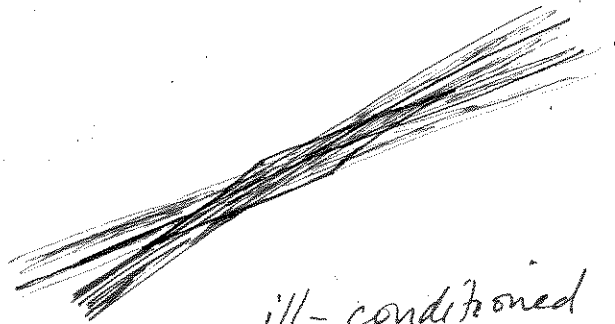
$$\frac{\|\Delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\Delta A\|}{\|A\|}$$

perturb b and A

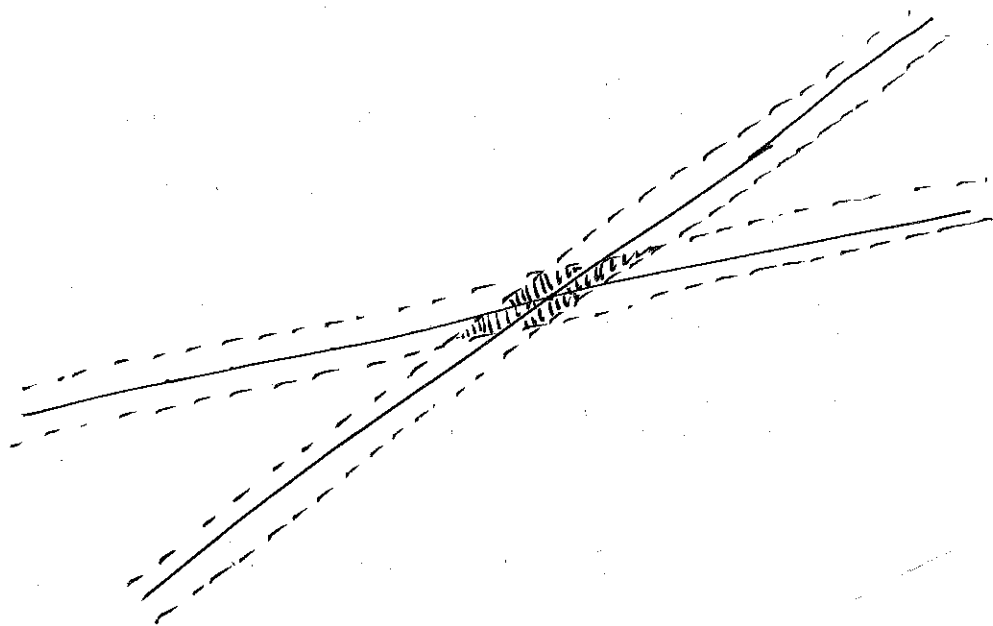
$$\frac{\|\Delta x\|}{\|x\|} \leq \kappa(A) \left(\frac{\|\Delta b\|}{\|b\|} + \frac{\|\Delta A\|}{\|A\|} \right)$$



well-conditioned



ill-conditioned



Summary: if input data ^(relatively) accurate to machine precision, then

$$\frac{\|\hat{x} - x\|}{\|x\|} \lesssim \text{cond}(A) \varepsilon_{\text{mach}}$$

computed solution loses up to

$$\log_{10}(\text{cond}(A))$$

digits of accuracy

Residual

$$r = b - A\hat{x}$$

$$\|r\| = 0 \iff \|\Delta x\| = 0$$

$$\text{relative residual} = \|r\| / (\|A\| \cdot \|\hat{x}\|)$$

$$\|\Delta x\| = \|\hat{x} - x\| = \|A^{-1}(A\hat{x} - b)\| = \|A^{-1}r\| \leq \|A^{-1}\| \|r\|$$

$$\frac{\|\Delta x\|}{\|\hat{x}\|} \leq \text{cond}(A) \frac{\|r\|}{\|A\| \cdot \|\hat{x}\|}$$

Residual

\hat{x} : approximate solution to
 $Ax = b$

$$r = b - A\hat{x}$$

For A nonsingular, $\|r\| = 0 \iff \|\Delta x\| = \|\hat{x} - x\| = 0$

But small residual alone does not imply small error

$$\begin{aligned} r &= b - A\hat{x} \\ &= Ax - A\hat{x} \\ &= A(x - \hat{x}) \\ &= -A\Delta x \end{aligned}$$

$$\Delta x = -A^{-1}r$$

$$\|\Delta x\| \leq \|A^{-1}\| \cdot \|r\|$$

$$\Rightarrow \frac{\|\Delta x\|}{\|\hat{x}\|} \leq \frac{\|A^{-1}\| \cdot \|A\| \cdot \|r\|}{\|A\| \cdot \|\hat{x}\|}$$

relative residual

Note: Scaling equations changes residual norm but not error.

\therefore use relative res.

$$\frac{\|r\|}{\|A\| \cdot \|\hat{x}\|}$$

$$\frac{\|\Delta x\|}{\|\hat{x}\|} \leq \text{cond}(A) \frac{\|r\|}{\|A\| \cdot \|\hat{x}\|}$$

small relative residual & well-conditioned A

\Rightarrow small relative error

What if ^{relative} residual is large?

Let E be st.

$$(A+E)\hat{x} = b$$

$$\|r\| = \|b - A\hat{x}\| = \|E\hat{x}\| \leq \|E\| \cdot \|\hat{x}\|$$

$$\Rightarrow \frac{\|r\|}{\|A\| \cdot \|\hat{x}\|} \leq \frac{\|E\|}{\|A\|}$$

large relative residual \Rightarrow large backward error

Ex. 2.8 Small Residual

$$Ax = \begin{bmatrix} .913 & .659 \\ .457 & .330 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .254 \\ .127 \end{bmatrix} = b$$

$$\hat{x}_1 = \begin{bmatrix} -0.0827 \\ 0.5 \end{bmatrix}, \quad \hat{x}_2 = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix}$$

$$\|r_1\|_1 = 2.1 \cdot 10^{-4}$$

$$\|r_2\|_1 = 2.4 \cdot 10^{-2}$$

exact $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

surprisingly \hat{x}_2 is better solution

(A is ill-conditioned.)