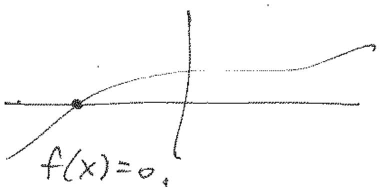
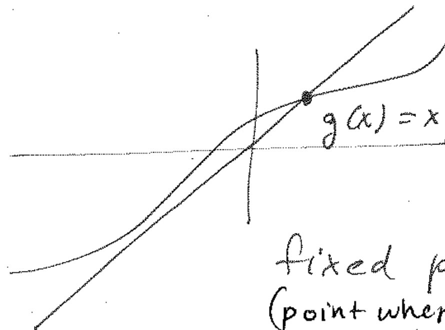
§ 5.5.2 Fixed Pt. Iteration

$x = g(x)$ $\qquad$ x is a "fixed pt"



root



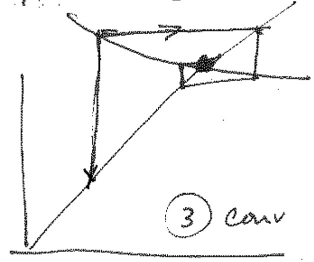
fixed point.
(point where curve intersects diagonal)

many choices of $g(x)$ for $f(x)=0$
— may differ in conv. rates & whether they converge at all!

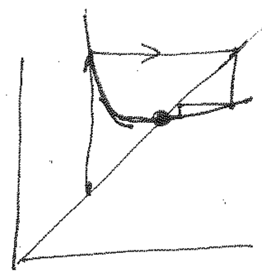
Example 5.8

$f(x) = x^2 - x - 2 = 0$ $\qquad$ soln: $x^* = 2$, $x^* = -1$

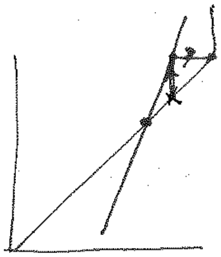
1. $g(x) = x^2 - 2$
2. $g(x) = \sqrt{x+2}$
3. $g(x) = 1 + 2/x$
4. $g(x) = \dfrac{(x^2+2)}{(2x-1)}$



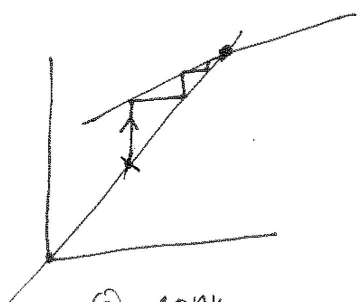
3 conv



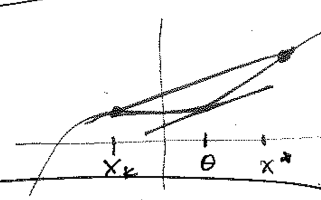
4 conv.



1 diverges



2 conv



locally convergent if $|g'(x^*)| < 1$

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$$

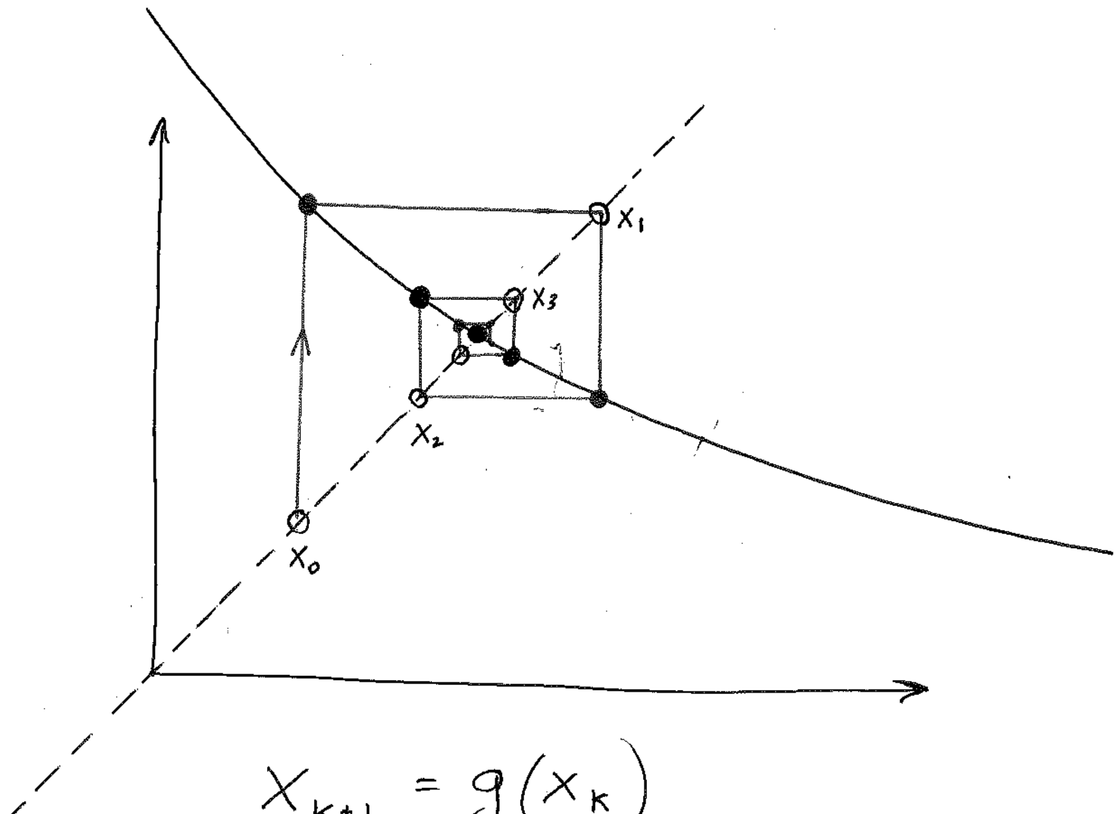
By Mean Value Theorem, $\exists$ θ between x_k & x^* s.t.

$$g'(\theta) = \frac{g(x_k) - g(x^*)}{x_k - x^*} \Rightarrow e_{k+1} = g'(\theta)\, e_k$$

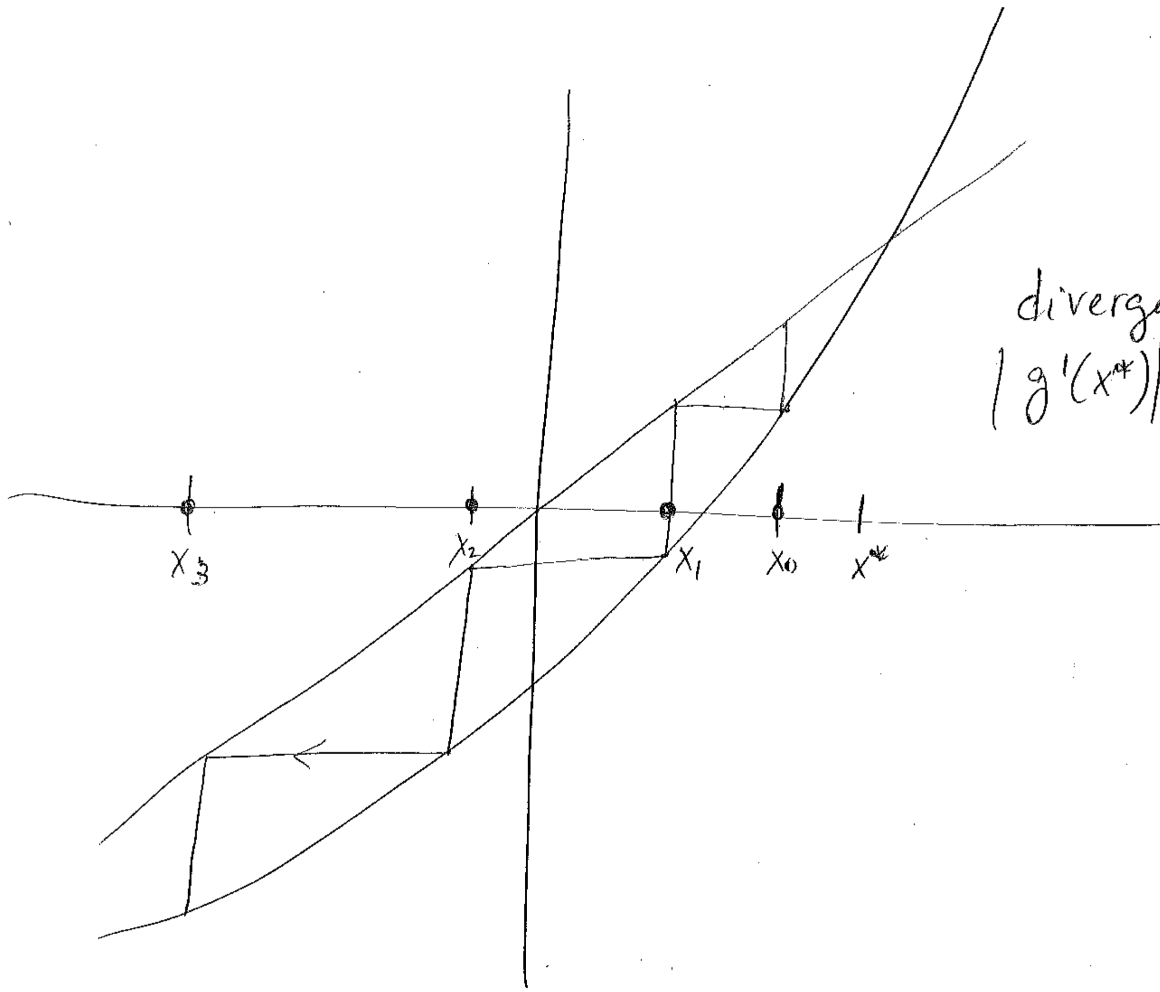
$$\Rightarrow |e_{k}| \le c^k |e_0|, \quad c < 1$$

$$\Rightarrow |e_k| \to 0$$

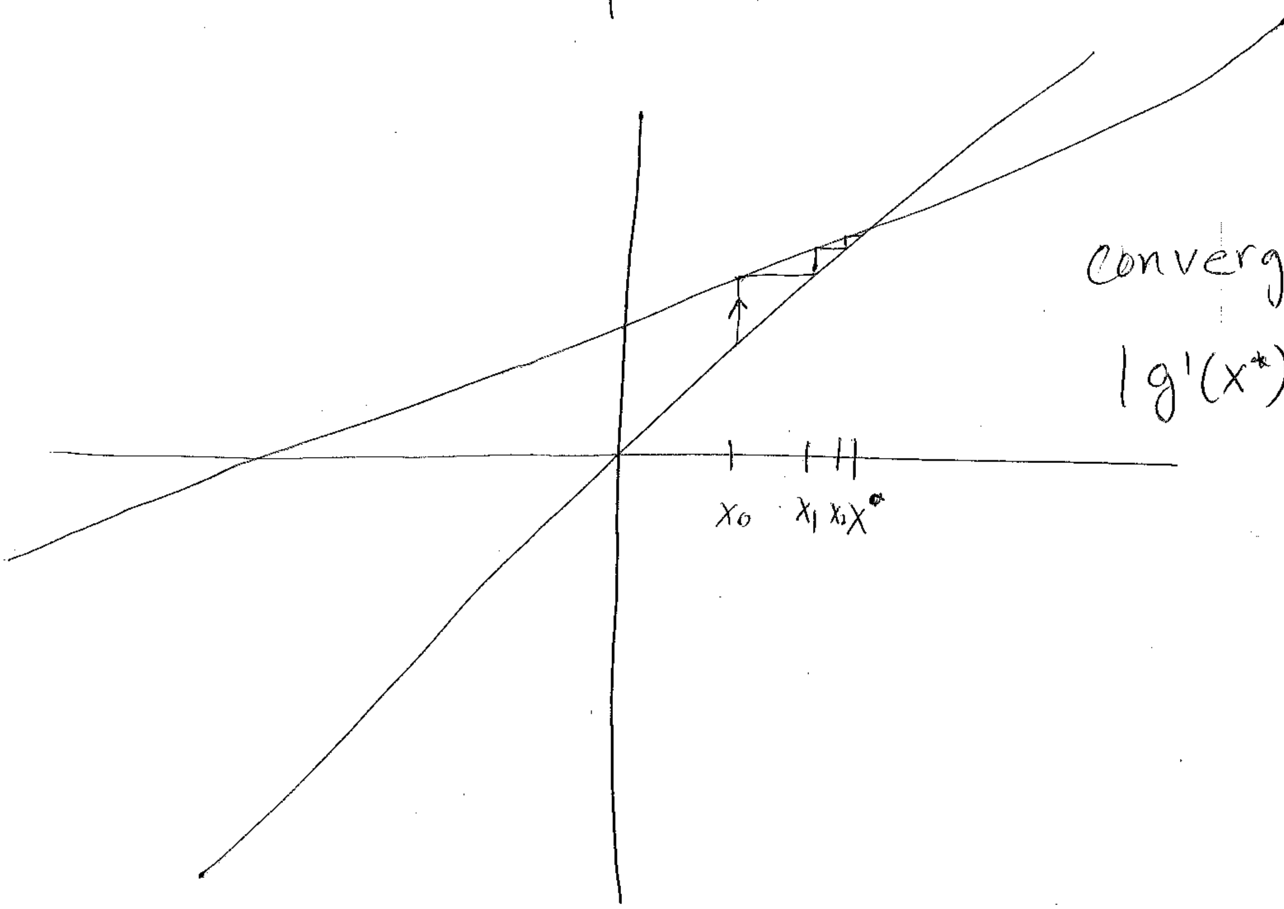
Fixed Point Iteration



$$x_{k+1} = g(x_k)$$



diverges
$|g'(x^*)| > 1$



converges
$|g'(x^*)| < 1$

Fixed Pt. Method Local Convergence

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - x^* = g(x_k) - g(x^*)$$

$$e_k = x_k - x^*$$

$$\boxed{\frac{e_{k+1}}{e_k} = \frac{g(x_k) - g(x^*)}{x_k - x^*}}$$

$$g(x_k) = g(x^*) + g'(x^*)(x_k - x^*) + \frac{1}{2}g''(x^*)(x_k - x^*)^2 + \dots$$

$$\Rightarrow g(x_k) - g(x^*) = g'(x^*)(x_k - x^*) + \frac{1}{2}g''(x^*)(x_k - x^*)^2 + \dots$$

$$\frac{e_{k+1}}{e_k} = g'(x^*)\frac{\cancel{(x_k - x^*)}}{\cancel{(x_k - x^*)}} + \frac{1}{2}g''(x^*)\frac{(x_k - x^*)^{\cancel{2}}}{\cancel{(x_k - x^*)}} + O((x_k - x^*)^2)$$

$$\boxed{\frac{e_{k+1}}{e_k} = g'(x^*) + \frac{1}{2}g''(x^*)(x_k - x^*) + O((x_k - x^*)^2)}$$

if $g'(x^*) = 0$,

$$\frac{e_{k+1}}{e_k} = \frac{1}{2}g''(x^*)(x_k - x^*) + \frac{1}{3!}g'''(x^*)(x_k - x^*)^2 + \dots$$

$$\Rightarrow \frac{e_{k+1}}{e_k^2} = \frac{1}{2}g''(x^*) + \frac{1}{3!}g'''(x^*)(x_k - x^*) + \dots$$

$$\boxed{\lim_{k \to \infty} \frac{e_{k+1}}{e_k^2} = \frac{1}{2}g''(x^*)}$$

quadratic convergence.

when fixed pt. conv., asymptotic conv rate is linear with $c = |g'(x^*)|$

Ideally have $g'(x^*) = 0$.

$$\Rightarrow \quad g(x_k) - g(x^*) = \frac{g''(\xi_k)}{2}(x_k - x^*)^2$$

$$\Rightarrow \quad \frac{g(x_k) - g(x^*)}{(x_k - x^*)^2} = \frac{g''(\xi_k)}{2} = \frac{e_{k+1}}{e_k^2}$$

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^2} = \frac{g''(x^*)}{2}$$

quadratic convergence.

Example 5.9

(1) $g'(x) = 2x$

$g'(2) = 4 \Rightarrow$ diverges

(2) $g'(x) = \frac{1}{2}(x+2)^{-1/2}$

$g'(2) = \frac{1}{2}(4)^{-1/2} = \frac{1}{4} \Rightarrow$ converges $c = \frac{1}{4}$

+tive sign $\Rightarrow$ iteration approaches from one side.

(3) $g'(x) = -2x^{-2}$

$g'(2) = -\frac{2}{2^2} = -\frac{1}{2} \Rightarrow$ converges $c = +\frac{1}{2}$

$-$ive sign $\Rightarrow$ spiral

(4) $g'(x) = \frac{2x^2 - 2x - 4}{(2x-1)^2} \Rightarrow$ converges quadratically

$g'(2) = 0$

$$g(x_k) = g(x^*) + \frac{(x_k - x^*)}{1} g'(x^*) + \frac{(x_k - x^*)^2}{2} g''(x^*) + \dots$$

Taylor's theorem

$$g(x_k) = g(x^*) + \cancelto{0}{(x_k - x^*) g'(x^*)} + \frac{(x_k - x^*)^2}{2} g''(\xi)$$

§ 5.5.3 Newton's Method

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

truncate

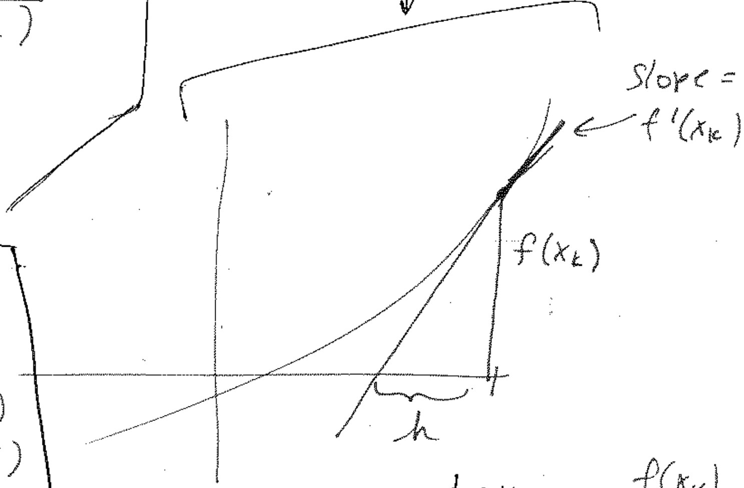
$$f(x+h) \cong f(x) + h f'(x)$$

$$\Rightarrow \quad h = \frac{f(x+h) - f(x)}{f'(x)}$$

want $f(x+h) = 0$

$$\Rightarrow \quad h = -\frac{f(x)}{f'(x)}$$

derivations of Newton's Method



$$f'(x_k) = \frac{f(x_k)}{h}$$

Algorithm

x_0 = initial guess
for $k = 1, 2, \ldots$
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$
end

transforms into a fixed pt iteration

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$= x - f(x)\left(f'(x)\right)^{-1}$$

$$g'(x) = 1 - f'(x)(f'(x))^{-1} + f(x)\, f'(x)^{-2} f''(x)$$

$$= \cancel{1 - 1} + \frac{f(x) f''(x)}{f'(x)^2}$$

if x^* is a simple root $\begin{cases} f(x) = 0 \\ f'(x) != 0 \end{cases}$ $\Rightarrow$ $g'(x) = 0$ $\Rightarrow$ quadratic convergence.

if x^* is a multiple root (multiplicity m) $\Rightarrow$ linear conv $C = 1 - \frac{1}{m}$ ~~mmul~~

See example 5.11