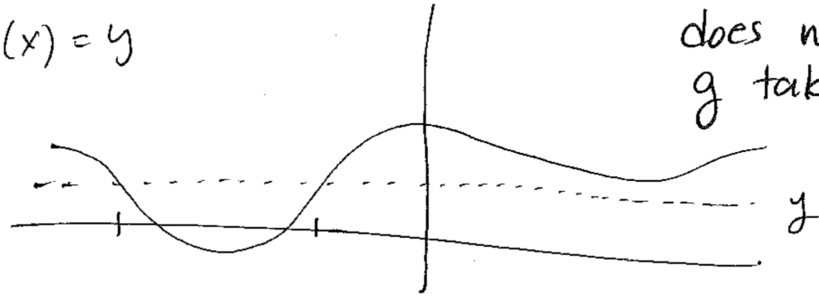
Nonlinear Equations

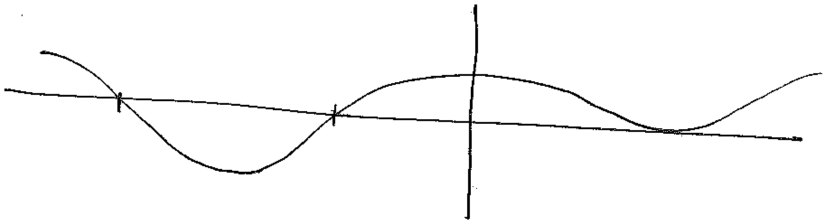
$g(x) = y$

For what values, x, does nonlinear function g take on value y?



$f(x) = g(x) - y = 0$

express as root finding problem



(*) $\vec{f}(\vec{x}) = \vec{0}$ $\qquad \vec{f} : \mathbb{R}^n \to \mathbb{R}^m$

m nonlinear equations in n unknowns

take $m = n$

$\vec{x}$ is a "root" of the equation or "zero" of function f

Solving (*) is <u>root finding</u> or <u>zero finding</u>.

Example 5.1

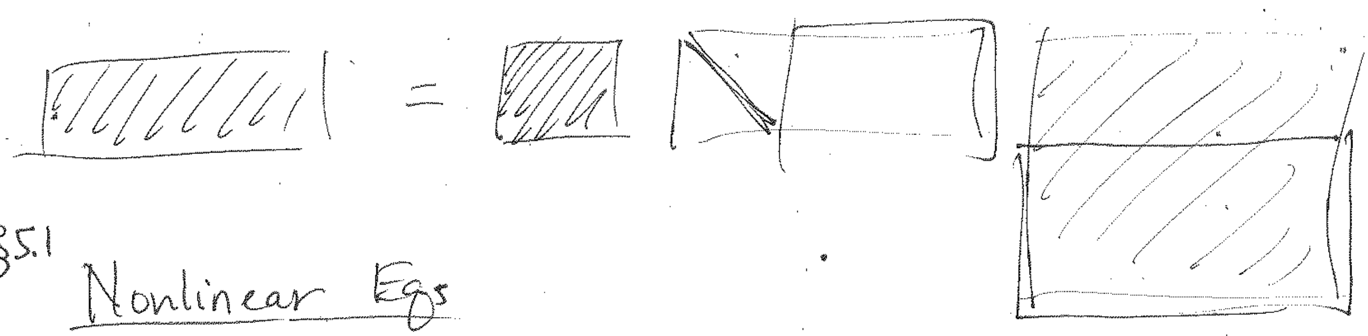
<u>1D</u> $\qquad f(x) = x^2 - 4\sin x = 0$

$\Rightarrow x \approx 1.93375$ one solution

<u>2D</u>

$$\vec{f}(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2 + .25 \\ -x_1 + x_2^2 + .25 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} .5 \\ .5 \end{pmatrix}$$



§5.1

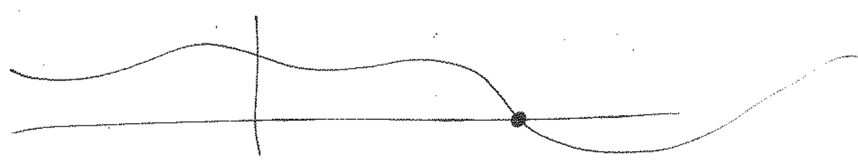
Nonlinear Eqs

more difficult than linear!

consider scalar case first.

$$g(x) = y$$
$$g(x) - y = 0$$
$$f(x) = 0$$

find "root" or "zero"
solution = root or zero

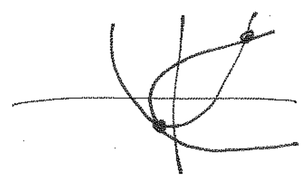


§ 5.2 Existence & Uniqueness

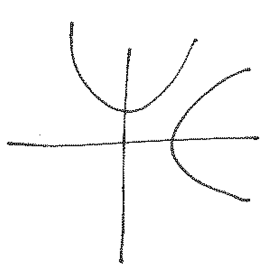
- nonlinear equations can have any # of solution
- each equation is a hypersurface in $\mathbb{R}^n$

Ex. 5.2

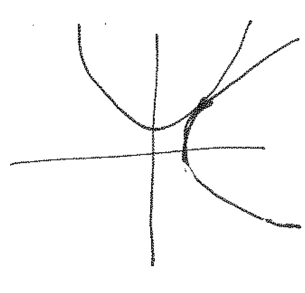
$$x^2 - y + \alpha = 0$$
$$-x + y^2 + \alpha = 0$$



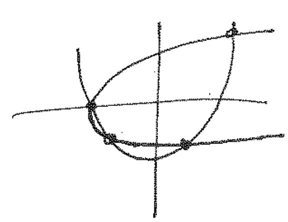
$\alpha = -\frac{1}{2}$
(2 solutions)



$\alpha = \frac{1}{2}$
(0 solutions)



$\alpha = \frac{1}{4}$
(1 solution)



$\alpha = -1$
(4 solutions)

Ex. 8.3

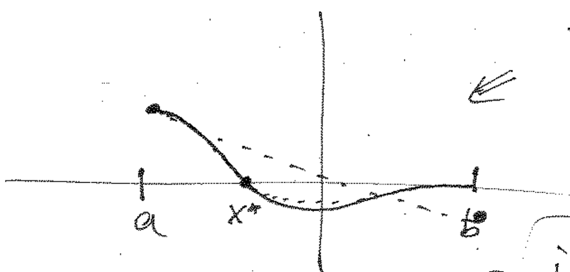
Lectures 8 & 9

~~Nonlinear equations~~

~~$f(x)=0$~~

	# solutions
$e^x+1=0$	0
$e^{-x}-x=0$	1
$x^2-4\sin x=0$	2
$x^3+6x^2+11x-6=0$	3
$\sin(x)=0$	inf

Existence of a zero – intermediate value theorem

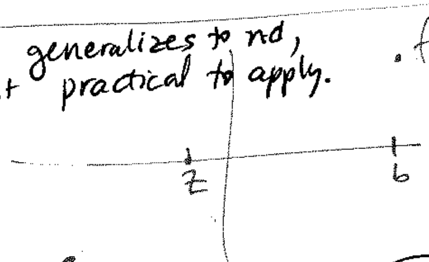


f continuous on $[a,b]$ $\Rightarrow \exists$ for any $c\in[f(b),f(a)]$ $\exists\, x^*$ s.t. $f(x^*)=c$

$\Rightarrow$ $\boxed{\text{if } f(a)\cdot f(b)<0, \text{ then there's a root in } [a,b]}$

$[a,b]$ is a "bracket" for solution of $f(x)=0$.

Above generalizes to nd, but not practical to apply.

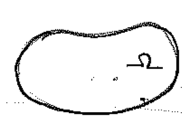


check $(b-z)\,f(b)\geq 0 \iff f(b)\geq 0$

$(a-z)\,f(a)\geq 0 \iff f(a)\leq 0$

another way of saying above $\rightarrow$ $\boxed{(x-z)\,f(x)\geq 0 \quad x=a,b}$

$\boxed{\text{generalization to multiple dim.}}$ $(x-z)^T f(x)\geq 0$ for x on $\partial\Omega$



Uniqueness? local

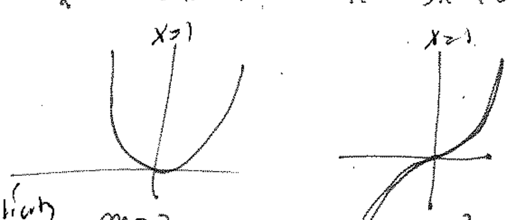
Simple root $f'(x^*)\neq 0$ ($\Leftrightarrow J_f(x^*)$ invertible)

multiple root $f'(x^*)=0$.

$f'(x^*)$ ~ cond. #.

$J_f(x^*)$

x^2-2x+1 $\qquad$ x^3-3x^2+3x-1



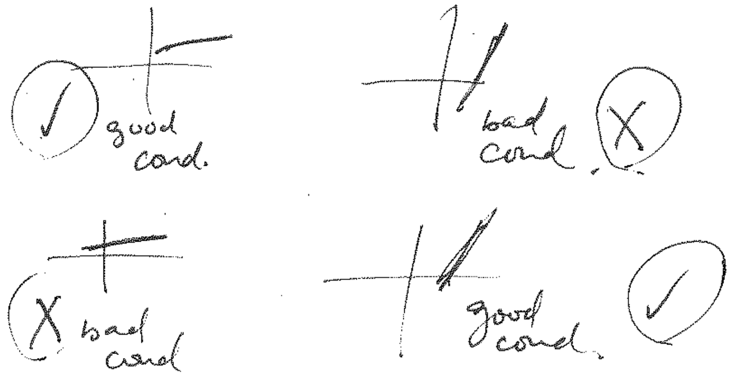
multiplicity $m=2$ $\qquad$ $m=3$.

§5.3 Sensitivity & Conditioning

opposite sensitivity

evaluation $\quad f(x) = y$

solution (root) $\quad f(x) = y$



cond # for eval $= |f'(x^*)|$

root $= \left|\dfrac{1}{f'(x^*)}\right|$

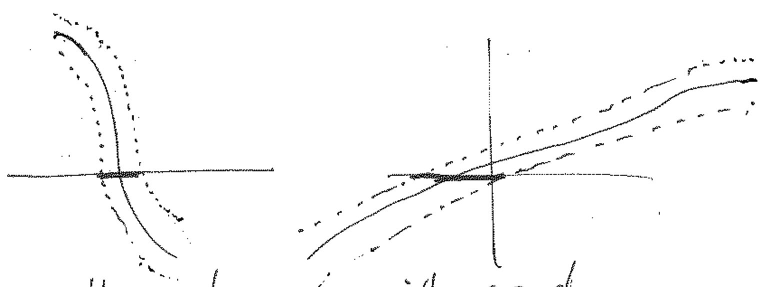
$$|\hat{x} - x^*| \le \left|\frac{1}{f'(x^*)}\right| \varepsilon$$

can be large if $f'(x^*)$ is small

mult.-D

eval : $\| J_f(x^*) \|$

root : $\| J_f^{-1}(x^*) \|$



well-cond $\qquad$ ill-cond.

residual $\quad \|f(\hat{x})\|$

error $\quad \|\hat{x} - x^*\|$

small resid $\Rightarrow$ small err only if cond # is small.

§5.4 Conv. Rates & Stopping Criteria.

Iterative methods (vs. direct methods).

$$\text{Cost} = \frac{\text{cost}}{\text{iter}} \cdot \#\text{iter}$$

Conv. rate

$e_k = x_k - x^*$ error at iter k.

conv w/ rate r if

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C, \quad C > 0.$$

- $r = 1$, $C < 1$ ⇒ linear — gain constant # of correct digits each iter $-(\log_\beta(C))$
- $r > 1$ ⇒ superlinear
- $r = 2$ ⇒ quadratic
- $r = 3$ ⇒ cubic

$r\times$ as many correct digits as in prev iter.

$\boxed{\text{plot } \log e^{k+1} \text{ vs } \log e^k, \text{ slope} = r}$

Ex. 5.6. Conv rates.

$10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, \ldots$ linear $C = 10^{-1}$

$10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}$ linear $C = 10^{-2}$

$10^{-2}, 10^{-3}, 10^{-5}, 10^{-8}, 10^{-12}$ superlinear, not quad.

$10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}$ quadratic.

Stopping criteria

don't know e_k — are difficult to give general

$\|x_{k+1} - x_k\| / \|x_k\|$

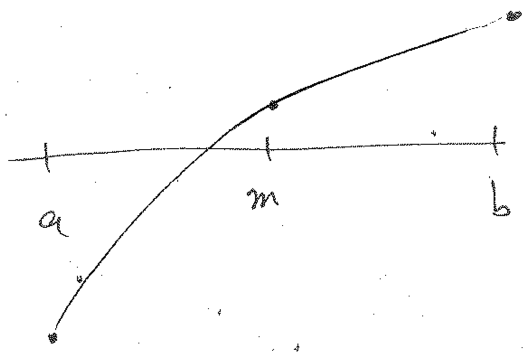
5.5 1D

cont. function $f: \mathbb{R} \to \mathbb{R}$

find $x^* \in \mathbb{R}$ s.t. $f(x^*) = 0$.

Bisection Method

find short interval $[a, b]$ where $f(x)$ changes sign



while $((b-a) > tol)$ do

  midpoint $= m = a + \left(\frac{b-a}{2}\right)$

  if $\text{sign}(f(a)) = \text{sign}(f(m))$

    $a = m$

  else

    $b = m$

  end

end

Numerical Pitfalls

~~midpoint = $m = \frac{a+b}{2}$~~

1 $[0.67, 0.69]$

$\boxed{\frac{a+b}{2}}$ $\quad 0.67 + 0.69 = 1.36$

vs. $\quad 2\overline{)1.4} = .7$ ✓ — result not in interval

$\boxed{a + \left(\frac{b-a}{2}\right)}$ $\quad \frac{.02}{2} = .01$

$0.67 + .01 = .68$

2 $a+b$ could overflow

3 $f(a) \cdot f(m) > 0$

as $f \to 0$, this could underflow.

$\boxed{\text{check sign explicitly}}$

- guaranteed to converge
- slow convergence.
  - linear convergence $(r = 1)$
  - $C = .5$ gain one bit of each iter
- length at iter k

  $\frac{b-a}{2^k}$

$\Rightarrow \frac{b-a}{2^k} < tol \Rightarrow \frac{b-a}{tol} < 2^k \Rightarrow \boxed{\log_2\left(\frac{b-a}{tol}\right) < k}$

See example 5.7