Tu, 10/23/2012   Lecture 7

4.1 Eigenvalues & Eigenvectors

A $n \times n$

$Ax = \lambda x$ right eigenvector "eigenvector"

$y^T A = \lambda y^T$ left eigenvector

$\Rightarrow A^T y = \lambda y$ (right eigv. of A^T).

– expand or shrink x by a scalar multiple.

Spectrum of A $= \lambda(A)$

$\rho(A) = \max\{|\lambda| : \lambda \in \lambda(A)\}$ spectral radius of A.

EXAMPLES

Diagonal Matrices: (eigv identity cols)

- $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Upper Triang: (and lower triangular: λ on diag)

- $\begin{pmatrix} 2 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (eigv not nec. identity cols).

Symmetric:

- $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Non-Symmetric:

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = i\begin{pmatrix} 1 \\ i \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = -i\begin{pmatrix} i \\ 1 \end{pmatrix}$

Complex matrices & vectors

replace A^T by A^H (or A^*)

transpose by conjugate transpose

4.2.1.

Characteristic Polynomials

$$Ax = \lambda x \iff (A - \lambda I)x = 0.$$

$$\iff \det(A - \lambda I) = 0$$

n - degree polynomial in λ.

roots are eigenval

Example

$$\det\left(\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$= (3-\lambda)^2 - 1 = 9 - 6\lambda + \lambda^2 - 1 = \lambda^2 - 6\lambda + 8$$

$$= (\lambda - 2)(\lambda - 4)$$

(Now find associated evecs)

Fundamental Theorem of Algebra

$$p(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \dots + c_n\lambda^n$$

$$= c_n(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

n roots (real or complex, distinct or repeated).

$\Rightarrow$ A $n \times n$ always has n eigenvalues.

A real

- λ real or ,
- λ occur in complex conjugate pairs

$$\lambda, \bar{\lambda} \qquad \lambda = \alpha + i\beta$$

But char. poly. not used as computational method:

- expensive to get the coeff in general.
- coeff. unstable to perturb. in matrix entries
- numerically sensitive to forming poly. roots
- expensive to find ~~rot~~ roots of polynomial.

e.g. $A = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$ $\quad \varepsilon^2 < \varepsilon_{mach}$. $\Rightarrow \lambda = 1+\varepsilon, 1-\varepsilon$

$$\det(A-\lambda I) = \lambda^2 - 2\lambda + 1 - \varepsilon^2 \approx \lambda^2 - 2\lambda + 1 \Rightarrow \lambda = 1, 1$$

4.2.2.

Algebraic Multiplicity

- root multiplicity

geo mult. < alg. mult $\Rightarrow \lambda$ "defective"

Geometric Multiplicity

- # of lin. indep eigenvec in space

Non-defective : $\Leftrightarrow$ diagonalizable.

$$AX = X\Lambda \Rightarrow A = X\Lambda X^{-1}$$
$$\Rightarrow X^{-1}AX = \Lambda$$

Similarity transformation ⟶

4.2.3

Eigenvectors can <u>be scaled</u>

$$Ax = \lambda x$$

$$A(\alpha x) = \lambda(\alpha x) \quad \checkmark.$$

usually take x s.t. $\|x\|_2 = 1$

$$S_\lambda = \{ x : Ax = \lambda x \} \quad \text{eigenspace}$$

"invariant subspace"

- $\dim(S_\lambda)$ = geometric mult. of λ.

Similarity (cont.)

A B <u>similar</u> if

$$A = X^{-1} B X \quad \text{for nonsingular } X$$

same <u>eigenvalues</u>
diff <u>eigenvectors</u>.

$$Av = \lambda v$$

$$\Rightarrow (X^{-1} B X) v = \lambda v$$

$$\Rightarrow B(Xv) = \lambda (Xv)$$

$\Rightarrow$ λ is an eigenvalue of B with eigenvector Xv

Symmetric Matrices (Real)

- guaranteed to have *real* eigenvalues.

Show:

$$Ax = \lambda x$$

$$x^H A x = \lambda x^H x = \lambda \|x\|_2^2$$

$$(x^H A x)^H = x^H A^H x = \lambda^H \|x\|_2^2$$

$$(A = A^H) \Rightarrow \quad x^H A x = \lambda^H \|x\|_2^2 = \lambda \|x\|_2^2$$

$$\Rightarrow \boxed{\lambda^H = \lambda}$$

- eigenvectors orthogonal:

$$\boxed{A = U \Lambda U^T}$$

similarly,

Hermitian Matrices (Complex)

- Λ real, U unitary

$$A = U \Lambda U^H$$

(same for "normal" matrices)

Note: Any square, complex matrix can be put into upper triangular form through unitary transformation

SCHUR DECOMPOSITION

$$A = UTU^T$$

eigenvalues can then be read off the diagonal.

Any Matrix can be put into Jordan form

$$\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$$ diagonal + 1 in super-diag. for defective eigenvalues.

e.g. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ vs. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

4.3. Sensitivity & Conditioning. $Ax = \lambda x$

- not the same as K for $Ax = b$
- not the same for diff. λ.

Can be shown that for $A \leftarrow A + E$

$$|\mu - \lambda_k| \le \text{cond}_2(X) \, \|E\|_2$$

so cond # = 1 for normal matrices

can be poor for ~~defective or~~ nearly defective matrices.

For single eigenvalue:

x, y right & left eigenvectors.

$$\Delta\lambda \le \frac{1}{|y^H x|} \|E\|_2$$

multiple or close eigenvalues can be poorly conditioned

"Balancing" (rescaling by diag. sim. transf. can be used to improve conditioning)