Tu, 10/23/2012 Lecture 7

4.1 Eigenvalues & Eigenvectors

A $n \times n$

$Ax = \lambda x$ right eigenvector "eigenvector"

$y^T A = \lambda y^T$ left eigenvector

$\Rightarrow A^T y = \lambda y$ (right eigv. of A^T).

– expand or shrink x by a scalar multiple.

Spectrum of A $= \lambda(A)$

$\rho(A) = \max\{|\lambda| : \lambda \in \lambda(A)\}$ spectral radius of A.

EXAMPLES

Diagonal Matrices : (eigv identity colns)

- $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$, $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{1\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

Upper Triang : (and lower triangular : $\underline{\lambda \text{ on diag}}$)

- $\begin{pmatrix} 2 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1\underline{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$ (eigr not nec. identity cols).

Symmetric :

- $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4\underline{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2\underline{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

Non-Symmetric :

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = i\begin{pmatrix} 1 \\ i \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = -i\begin{pmatrix} i \\ 1 \end{pmatrix}$

Complex matrices & vectors

replace $\cdot^T$ by $\cdot^H$
transpose by conjugate transpose

4.2.1.
Characteristic Polynomials

$$Ax = \lambda x \iff (A - \lambda I)x = 0.$$

$$\iff \det(A - \lambda I) = 0$$

~~high d~~ n - degree polynomial in λ.

roots are eigenval

Example.

$$\det\left(\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$= (3-\lambda)^2 - 1 = 9 - 6\lambda + \lambda^2 - 1 = \lambda^2 - 6\lambda + 8$$
$$= (\lambda - 2)(\lambda - 4)$$

Fundamental Theorem of Algebra

$$p(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \dots + c_n\lambda^n$$

$$= c_n(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

n roots (real or complex, distinct or repeated).

$\Rightarrow$ A $n \times n$ always has $\underline{n}$ eigenvalues.

A real

- λ real or
- λ occur in complex conjugate pairs

$\lambda, \bar{\lambda}$ $\qquad$ $\lambda = \alpha + i\beta$

But char. poly. <u>not</u> used as computational method:

- expensive to get the coeff in general.
- coeff. unstable to perturb. in matrix entries
- numerically sensitive to forming poly.
- expensive to find ~~tot~~ roots of polynomial. (λ roots)

e.g. $A = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$ $\quad \varepsilon < \sqrt{\varepsilon_{mach}}$. $\quad \Rightarrow \lambda = 1+\varepsilon, 1-\varepsilon$

$\det(A-\lambda I) = \lambda^2 - 2\lambda + 1 - \varepsilon^2 \approx \lambda^2 - 2\lambda + 1$

$\Rightarrow \lambda = 1, 1$

4.2.2.

<u>Algebraic Multiplicity</u>

- root multiplicity

geo mult. < alg. mult $\Rightarrow \lambda$ "defective"

<u>Geometric Multiplicity</u>

- # of lin. indep eigenvec in space

<u>Non-defective</u>: $\Leftrightarrow$ ~~diag~~ diagonalizable.

$AX = X\Lambda \Rightarrow A = X\Lambda X^{-1}$

$\Rightarrow X^{-1}AX = \Lambda$

Similarity <u>transformation</u>

4.2.3

Eigenvectors can be scaled

$$Ax = \lambda x$$

$$A(\alpha x) = \lambda(\alpha x) \quad ✓.$$

usually take x s.t. $\|x\|_2 = 1$

$S_\lambda = \{ x : Ax = \lambda x \}$ eigenspace

"invariant subspace"

- $\dim(S_\lambda)$ = geometric mult. of λ.

Similarity (cont.)

A B similar if

$A = X^{-1} B X$ for nonsingular X

$Av = \lambda v$

same eigenvalues

diff eigenvectors.

$\Rightarrow (X^{-1} B X) v = \lambda v$

$\Rightarrow B(Xv) = \lambda (Xv)$

$\Rightarrow$ λ is an eigenvalue of B with eigenvector Xv

Symmetric Matrices (Real)

- guaranteed to have real eigenvalues.

Show:

$$Ax = \lambda x$$

$$x^H A x = \lambda x^H x = \lambda \|x\|_2^2$$

$$(x^H A x)^H = x^H A^H x = \lambda^H \|x\|_2^2$$

$$(A = A^H) \Rightarrow \quad x^H A x = \lambda^H \|x\|_2^2 = \lambda \|x\|_2^2$$

$$\Rightarrow \boxed{\lambda^H = \lambda}$$

- eigenvectors orthogonal:

$$\boxed{A = U \Lambda U^T}$$

similarly,

Hermitian Matrices (complex)

$$A = U \Lambda U^H$$

(same for "normal" matrices)

Note: Any matrix can be put into upper triangular form through unitary transformation

$$A = U T U^T$$

eigenvalues can then be read off the diagonal.

Any Matrix can be put into <u>Jordan form</u>

$$\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$$

diagonal + 1 in super-diag. for <u>defective</u> eigenvalues.

4.3. Sensitivity & Conditioning

$Ax = \lambda x$

- <u>not</u> the same as K for $Ax = b$
- <u>not</u> the same for diff. λ.

Can be shown that for $A \leftarrow A + E$

$$|\mu - \lambda_k| \leq \text{cond}_2(X) \, \|E\|_2$$

so cond # = 1 for normal matrices

can be poor for ~~defective or~~ nearly defective matrices.

<u>For single eigenvalue:</u>

x, y right & left eigenvectors.

$$\boxed{\Delta\lambda \leq \frac{1}{|y^H x|} \|E\|_2}$$

multiple or close eigenvalues can be poorly conditioned

"Balancing" (rescaling by diag. sim. transf. can be used to improve conditioning)

4.4. Problem Transformations

Shifts

$$\begin{array}{rl} Ax &= \lambda x \\ -\ \sigma x &= -\sigma x \\ \hline (A-\sigma I)x &= (\lambda-\sigma)x \end{array}$$

eigenvalues <u>shifted</u>
eigenvectors unaffected.

Inversion.

$$Ax = \lambda x \qquad A \text{ nonsingular},$$

$$\Rightarrow \quad x = \lambda A^{-1}x$$

$$\Rightarrow \quad A^{-1}x = \frac{1}{\lambda}x$$

eigenvalues <u>reciprocal</u>
eigenvectors <u>unaffected</u>

Powers

$$Ax = \lambda x$$

$$A^2x = A(Ax) = A\lambda x = \lambda Ax = \lambda^2 x$$

eigenvalues <u>squared</u>
eigenvectors unaffected

Polynomials.

$$p(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_n t^n$$

$$p(A) = c_0 + c_1 A + c_2 A^2 + \dots + c_n A^n$$

$$Ax = \lambda x$$

$$p(A)x = p(\lambda)x$$

eigenvalues <u>$p(\lambda)$</u>
eigenvectors unaffected.

Similarity.

A B similar, $\exists T$ nonsingular

$$B = T^{-1}AT$$

$$By = \lambda y \Rightarrow T^{-1}ATy = \lambda y$$

$$\Rightarrow A(Ty) = \lambda(Ty)$$

eigenvalues unaffected
eigenvectors <u>transformed by T</u>

Similary Example

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

Jordan form

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

<u>not diagonalizable</u>.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} x + y = x \\ y = y \end{array} \Rightarrow \begin{array}{l} y = 0 \\ x = \alpha \end{array}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

Schur Form.

$$A = U T U^T$$

4.5. Computing Eigenvalues & Eigenvectors.

4.5.1 Power Iteration

$$x_k = A x_{k-1}$$

$x_k \rightarrow$ eigenvector corresponding to largest eigenvalue.

$$x_0 = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$$

$$A^k x_0 = \alpha_1 \lambda_1^k x_1 + \alpha_2 \lambda_2^k x_2 + \dots + \alpha_n \lambda_n^k x_n$$

$$= \lambda_1^k \left(\alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i \underbrace{\left(\frac{\lambda_i}{\lambda_1}\right)^k}_{\left(\frac{\lambda_i}{\lambda_1} < 1\right) \Rightarrow 0 \text{ as } k \to \infty.} x_i \right)$$

$$\rightarrow \lambda_1^k \alpha_1 x_1$$

<u>May fail:</u>

- $\alpha_1 = 0$.

unlikely; in practice, rounding error will introduce such a component.

- repeated λ_1

  $\rightsquigarrow$ converge to vector in subspace associated with λ_1

- for real A, real x_0 can't get complex value.

$\rightarrow$ rescale x_k to avoid underflow & overflow.

$\rightarrow$ convergence rate depends on $\left|\frac{\lambda_2}{\lambda_1}\right|$ $\left(> \left|\frac{\lambda_2 - \sigma}{\lambda_1 - \sigma}\right| \right)$ shift:

- can find extreme eigenvalues λ_1, λ_n

Inverse Iteration

$$x^{k+1} = A^{-1} x^k$$

i.e. $A x^{k+1} = x^k$

$x^{k+1} \leftarrow \dfrac{x^{k+1}}{\|x^{k+1}\|_\infty}$ normalize..

- can find any eigenvalue w/ appropriate shift,
- rapid convergence for $A - \tilde{\lambda} I$

4.5.4. Deflation

remove known eigenvalue.

e.g. $u_1^T x_1 = \lambda_1$

$(A - x_1 u_1^T)$ has eigenvalues $0, \lambda_2, \ldots, \lambda_n$

analogy:

$p(\lambda)$

$\dfrac{p(\lambda)}{\lambda - \lambda_1}$ removes known root

check: $(A - x_1 u_1^T) x_1 = A x_1 - x_1 (u_1^T x_1) = \lambda_1 x_1 - \lambda_1 x_1 = 0.$

$(A - x_1 u_1^T) x_2 = A x_2 - x_1 (u_1^T x_2) = \lambda_2 x_2 - x_1 u_1^T x_2$

$= (\lambda_2 I - x_1 u_1^T) x_2 \Leftarrow u_1 \perp x_2$ ✓.

---

combine w/ inverse iteration

Rayleigh Quotient

$A x = \lambda x$

$x^T A x = \lambda x^T x \longrightarrow \lambda = \dfrac{x^T A x}{x^T x}$

(cf least squares $x \lambda \cong A x$)

4.5.5 Simultaneous Iteration

$$X_{k+1} = A X_k$$

$$X_{k+1} = Q_{k+1} R_{k+1}$$

$$X_{k+2} = A Q_{k+1}$$

$$\vdots$$

$$\boxed{\begin{aligned} X_k &= Q_k R_k \\ X_{k+1} &= A Q_k \end{aligned}}$$

4.5.6 QR Iteration

$$A_0 = A$$

$$\begin{cases} Q_1 R_1 = A_0 \\ A_1 = R_1 Q_1 \quad (= Q_1^H A_0 Q_1) \end{cases}$$

$$\begin{cases} Q_2 R_2 = A_1 \\ A_2 = R_2 Q_2 \quad (= Q_2^H A_1 Q_2 = Q_2^H Q_1^H A_0 Q_1 Q_2) \end{cases}$$