Orthogonal Vectors and Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$$

transpose

and $A^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{32} \end{bmatrix}$

adjoint

Hermitian

$A = A^*$

Symmetric

$A \in R^{n \times n}, \quad A = A^T$

Real case:

<u>orthogonal vectors</u>, $\vec{x}, \vec{y}$

$$\vec{x}^T \vec{y} = 0$$

<u>orthonormal</u>

orthogonal & normalized

orthonormal set $\{\vec{q}_1, \dots, \vec{q}_n\}$ $\quad q_i \in \mathbb{R}^m$

Let $\vec{v} \in \mathbb{R}^m$

$\vec{r} = \vec{v} - (q_1^T v) q_1 - (q_2^T v) q_2 - \dots - (q_n^T v) q_n$

$\vec{q}_i^T \vec{r} = 0 \qquad$ (r orthogonal to q_i)

for $m = n$ decompose $\vec{v}$ as

$$\vec{V} = \sum_{i=1}^{n} (\vec{q}_i^T \vec{v}) \vec{q}_i = \sum_{i=1}^{n} (\vec{q}_i \vec{q}_i^T) \vec{V}$$

Orthogonal Matrix

$Q \in \mathbb{R}^{n \times n}$ is orthogonal if

$Q^T Q = I$

$\left(\begin{array}{l} Q \in \mathbb{C}^{n \times n}, \ \underline{\text{unitary}} \\ Q^* Q = I \end{array} \right)$

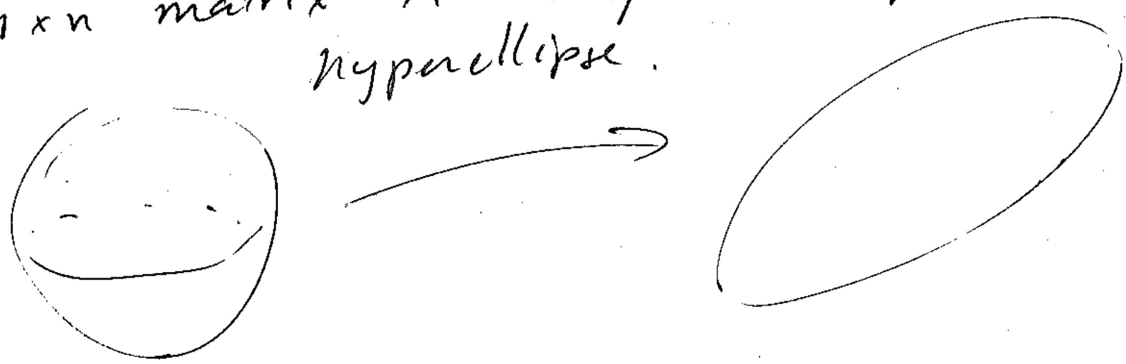
$$\begin{bmatrix} — & \vec{q}_1^{\,T} & — \\ — & \vec{q}_2^{\,T} & — \\ & \vdots & \\ — & \vec{q}_n^{\,T} & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

mult by. Q : rotation ($\det = 1$), or
reflection ($\det = -1$)

SVD

T&B: "Many problems of linear algebra can be better understood if we first ask the question: what if we take the SVD?"

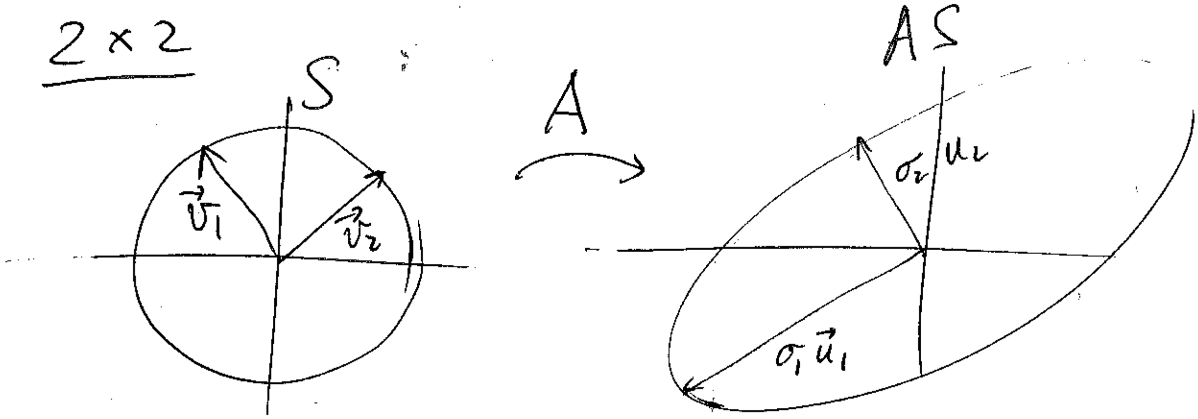
$n \times n$ matrix A maps unit sphere to hyperellipse.



hyperellipse: stretch sphere by factors $\sigma_1, \dots, \sigma_n$ in some orthogonal directions $\vec{u}_1, \dots, \vec{u}_n$

$\sigma_i \vec{u}_i$: principal semiaxes of the hyperellipse. lengths $\sigma_1, \dots, \sigma_n$

2 × 2



Singular Values the lengths $\sigma_1, \dots, \sigma_n$ of the n principal semiaxes of ellipse. AS

Left Singular Vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ in directions of principal semiaxes of AS

Right Singular Vectors $\vec{v}_1, \dots, \vec{v}_n$ preimages of $\sigma_j \vec{u}_j$ $\quad A\vec{v}_j = \sigma_j \vec{u}_j$

Reduced SVD

A $m \times n$

$$A v_j = \sigma_j u_j \qquad 1 \le j \le n$$

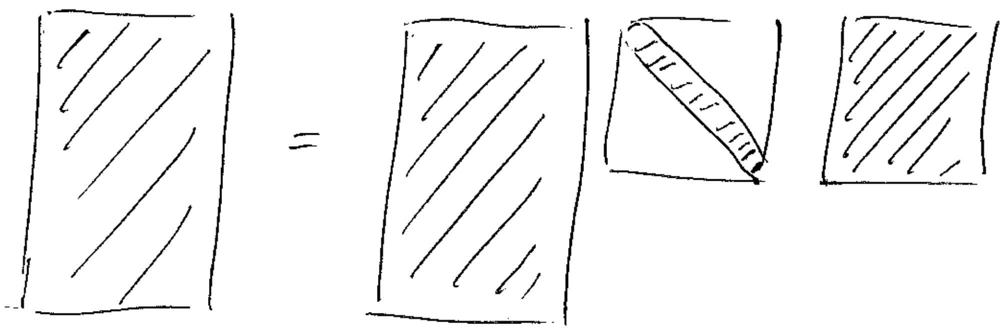
or

$$\underset{m \times n}{\begin{bmatrix} \\ A \\ \\ \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}} = \underset{m \times n}{\begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \\ | & | & & | \end{bmatrix}} \underset{n \times n}{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}}$$

$$AV = \hat{U}\hat{\Sigma}$$

$V, \hat{U}$ orthonormal columns

V orthogonal matrix

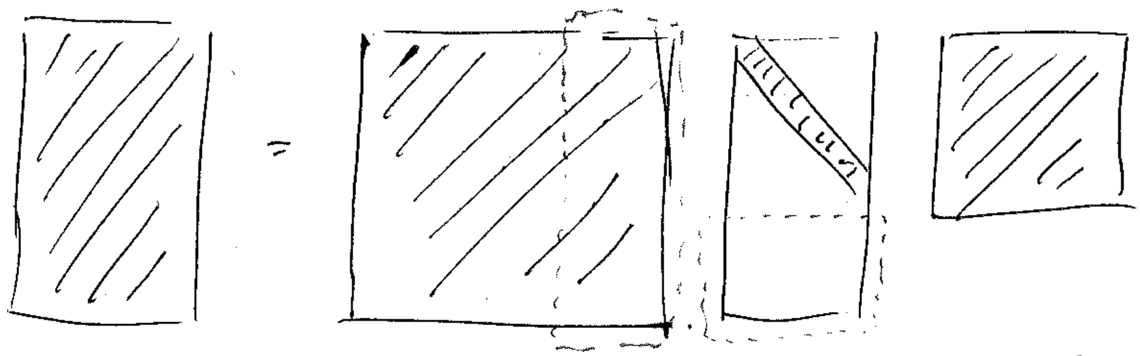


$$A = \hat{U} \quad \hat{\Sigma} \quad V^T$$

or $\quad (A = \hat{U}\hat{\Sigma}V^*)$

Full SVD

columns of U are n orthonormal vectors in m-dim space. If $n < m$, find $m-n$ additional orthonormal vectors to complete basis for $\mathbb{R}^m$ extend $\hat{U}$ to orthogonal matrix U.



$$\underset{m \times n}{A} \quad \underset{m \times m}{U} \quad \underset{m \times n}{\Sigma} \quad \underset{n \times n}{V^T}$$

Rank-deficient A

A $m \times n$, rank $r < n$

$$\underbrace{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r}_{\text{non-zero}} > \underbrace{\sigma_{r+1} = \cdots = \sigma_n}_{\text{zero}} = 0$$

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

$$\underset{m \times n}{A} = \underset{m \times m}{U} \underset{m \times n}{\Sigma} \underset{n \times n}{V^*} \qquad A \in \mathbb{C}^{m \times n}$$

Every A has svd. σ_i uniquely determined

(A square, σ_i distinct $\Rightarrow$ u_i, v_i unique up to complex sign)

Projector

projector: square matrix that satisfies

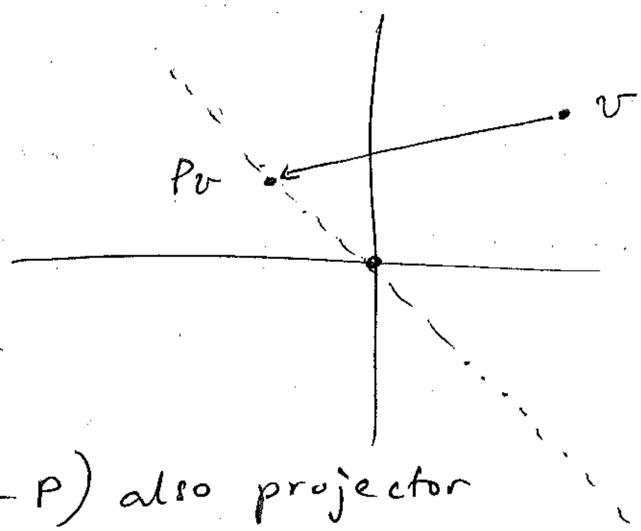
$$P^2 = P \qquad \text{(idempotent)}$$

orthogonal projector + non-orthogonal or projector
oblique

$v \in \text{Range}(P)$

$v = Px \quad \Rightarrow \quad Pv = P^2x = Px = v$

$$P(Pv - v) = P^2v - Pv$$
$$= Pv - Pv = 0$$



oblique projection

Complementary Projector

P projector $\Rightarrow (I - P)$ also projector

$$(I-P)^2 = I - 2P + P^2 = I - 2P + P$$
$$= I - P \quad \checkmark$$

Orthogonal Projector

(not an orthogonal matrix)

range$(P) \perp$ range$(I-P)$

Theorem Projector P is orthogonal projector iff $P = P^T$

Proof show $Px \perp (I-P)y \quad \forall x, y \quad \Leftrightarrow \quad P = P^T$ (2)

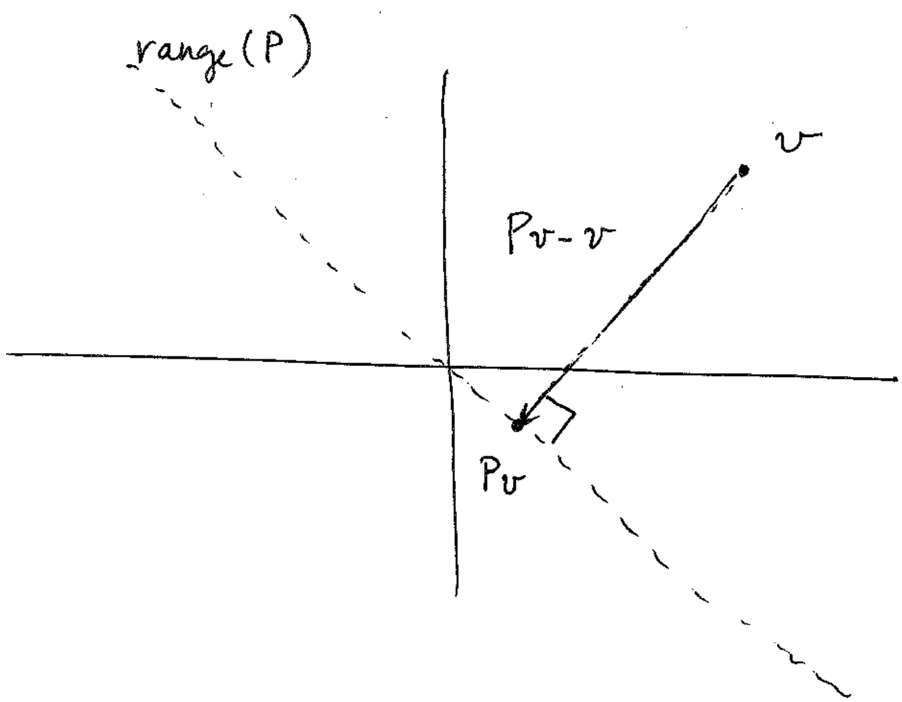
(1) $(Px)^T[(I-P)y] = x^T P^T (y - Py) = x^T P^T y - x^T P^T P y = 0$

Assume (2). Then
$$x^T P^T y - x^T P^T P y$$
$$= x^T P y - x^T P^2 y$$
$$= x^T P y - x^T P y \quad = 0 \Rightarrow (1) \checkmark$$

Assume (1). Then $x^T P^T y = x^T P^T P y \quad \forall x, y$

In particular $e_i^T P^T e_j = e_i^T P^T P e_j \quad \forall i, j \in \{1, \dots n\}$

$$\left.\begin{aligned} \Rightarrow \quad & P^T = P^T P \\ \text{taking transpose} \quad & P = P^T P \end{aligned}\right\} \Rightarrow P^T = P \Rightarrow (2) \checkmark$$



orthogonal projection

Projection with an orthonormal basis.

orthogonal projector P $(= P^T)$

$$P = U \Sigma U^T$$

and $P^2 = U\Sigma U^T U \Sigma U^T = U \Sigma^2 U^T$

$$\Rightarrow \quad \Sigma = \Sigma^2$$

singular values are all 0 or 1.

$(r = \operatorname{rank} P \le n)$ $P = \sum_{i=0}^{r} \vec{u}_i \vec{u}_i^T = \hat{U}\hat{U}^T$, where $\hat{U} = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{bmatrix}$

For any $\hat{Q}$ w/ orthonormal columns

$\hat{Q}\hat{Q}^T$ is an orthogonal projector onto columnspace of $\hat{Q}$.

Complementary projector $I - \hat{Q}\hat{Q}^T$ is also orthogonal projector

rank 1 $P = (\vec{q}\vec{q}^T)$ | $\vec{q}$ unit vector

rank $n-1$ $P^\perp = (I - \vec{q}\vec{q}^T)$

or $P = \dfrac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}}$ $\vec{v}$ arbitrary vector

normalize