# Vector Norms (Strang I.11)

All vector norms satisfy :

① $\|\vec{v}\| > 0$ when $\vec{v} \neq \vec{0}$, and
$\|\vec{v}\| = 0$ when $\vec{v} = \vec{0}$

② $\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$ scaling

③ $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ triangle inequality

## 3 important norms:

$\ell^2$ norm (Euclidean norm)

$$\|\vec{v}\|_2 = \left(|v_1|^2 + \dots + |v_n|^2\right)^{1/2}$$

$\ell^1$ norm (or 1-norm)

$$\|\vec{v}\|_1 = |v_1| + |v_2| + \dots + |v_n|$$

$\ell^\infty$ norm

$$\|\vec{v}\|_\infty = \max_{k=1,\dots,n} |v_k|$$

E.g. let $\vec{v} = (1, 1, \dots, 1)^T$. What are the 1-, 2-, and $\infty$- norms of $\vec{v}$?

$$\|\vec{v}\|_1 = 1 + 1 + \dots + 1 = n$$

$$\|\vec{v}\|_2 = (1 + 1 + \dots + 1)^{1/2} = n^{1/2} = \sqrt{n}$$

$$\|\vec{v}\|_\infty = \max_{k=1,\dots,n} 1 = 1$$

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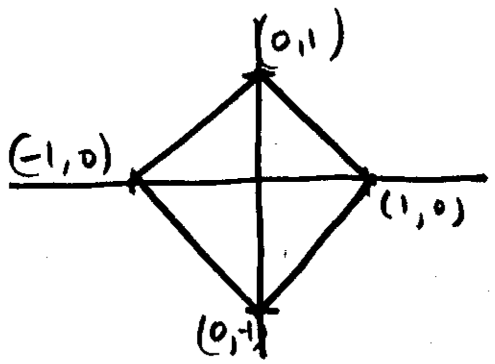
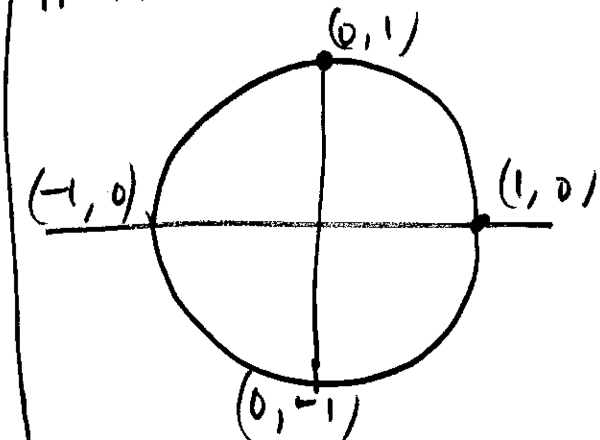
Let's draw the set of vectors $\vec{v}$ with $\|\vec{v}\| = 1$ for each of the three cases, in two dimensions.

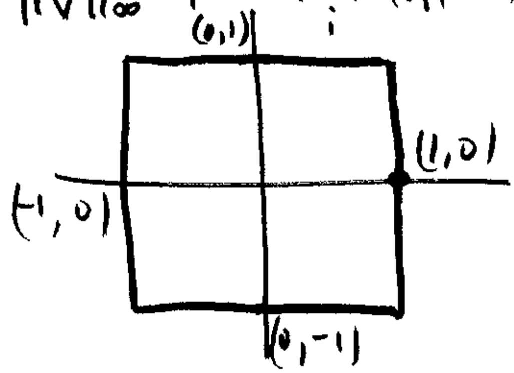
$\|\vec{v}\|_1 = 1 \qquad |v_1| + |v_2| = 1$




$\|\vec{v}\|_2 = 1 \qquad v_1^2 + v_2^2 = 1$




$\|\vec{v}\|_\infty = 1 \qquad \max_i |v_i| = 1$

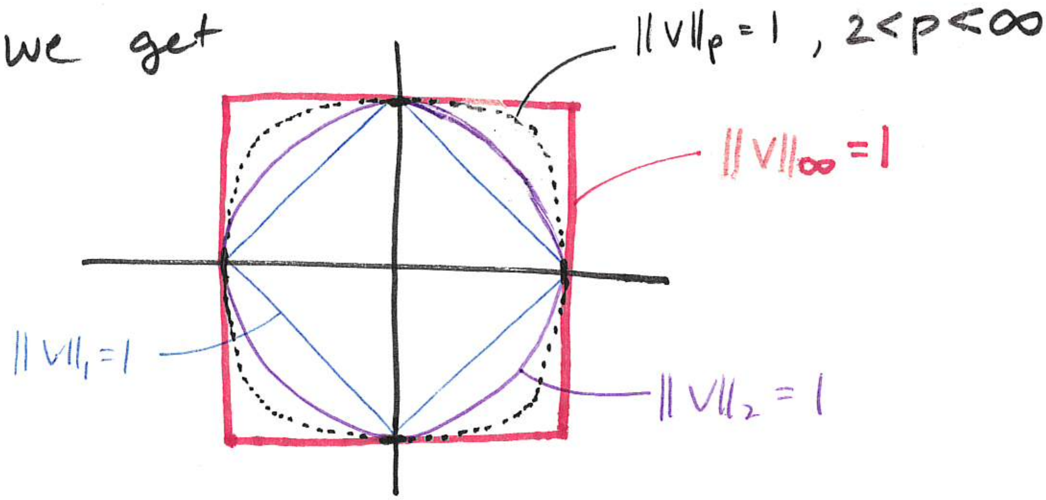


These are all examples of "p-norms"

$$\|\vec{v}\|_p = \left(|v_1|^p + |v_2|^p + \dots + |v_n|^p\right)^{\frac{1}{p}}$$

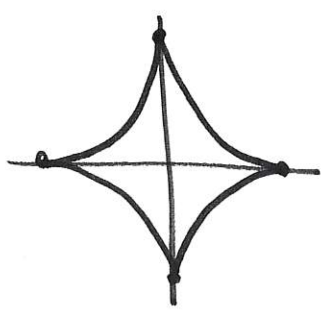
This a valid norm for $1 \le p \le \infty$.

If we combine the previous sketches for $\|v\|_1$, we get




For $p<1$, we don't get a norm, as the properties ①-③ of norms are not all satisfied. E.g.

$p = \frac{1}{2}$, $\|\vec{v}\|_{\frac{1}{2}} = 1$



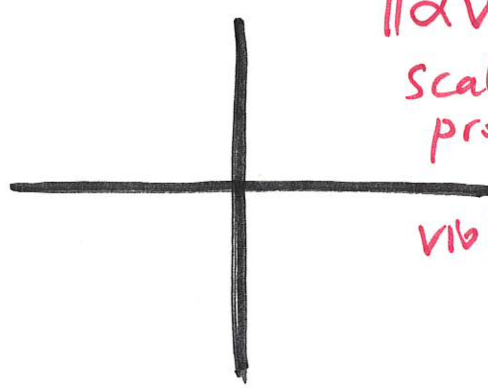
triangle ineq. violated

$$\underbrace{\left\|\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}\right\|_{\frac{1}{2}}}_{=4} \not\le \underbrace{\left\|\begin{pmatrix}0\\1\end{pmatrix}\right\|_{1/2}}_{1} + \left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|_{1/2}$$

$$4 \not\le 1 + 1$$

$p = 0$ (count of non-zero entries)

$\|\vec{v}\|_0 = 1$



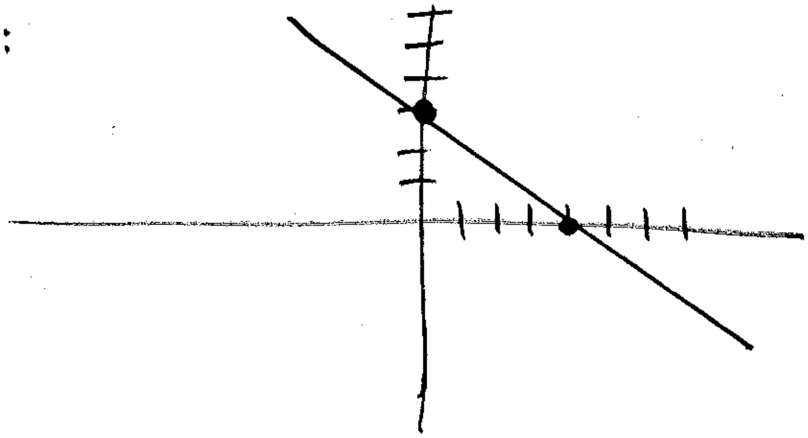
$\|\alpha\vec{v}\|_0 \ne \alpha\|\vec{v}\|_0$

Scaling property violated.

# The minimum of $\|\vec{v}\|_p$ on the line $a_1 v_1 + a_2 v_2 = 1$

Consider the line $3v_1 + 4v_2 = 1$.

Plot:

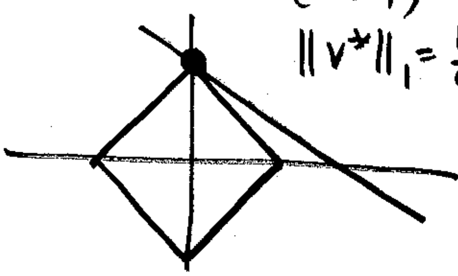


The minimum depends on which $p$ is used:

$\boxed{p=1}$

$(0, \frac{1}{4})$

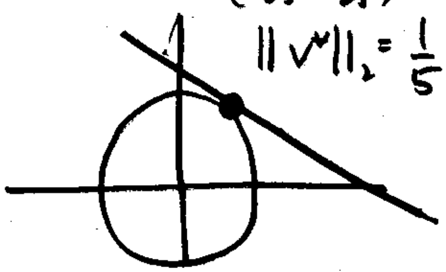
$\|v^*\|_1 = \frac{1}{4}$



$\boxed{p=2}$

$(\frac{3}{25}, \frac{4}{25})$

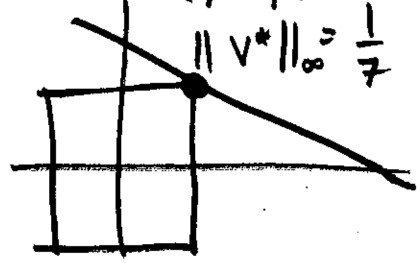
$\|v^*\|_2 = \frac{1}{5}$



$\boxed{p=\infty}$

$(\frac{1}{7}, \frac{1}{7})$

$\|v^*\|_\infty = \frac{1}{7}$



Notably, the solution in the 1-norm is <u>sparse</u>.
It has components equal to 0.
This is related to problems of "basis pursuit".

$\ell^0$ is not a norm. But a sparse solution to $Av=b$ can be found with the $\ell^1$-norm.

## Important properties of the 2-norm

$\vec{v} \cdot \vec{v} = \vec{v}^T \vec{v} = \|\vec{v}\|_2^2$ length squared is inner product

$\vec{v}^T \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$ angle $\theta$ between $\vec{v}$ and $\vec{w}$

## Important Inequalities:

Triangle inequality: $\|u+w\|_2 \le \|u\|_2 + \|w\|_2$

Cauchy-Schwartz inequality: $|v^T w| \le \|v\|_2 \|w\|_2$

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For any symmetric, positive definite matrix $S$, we can define the S-norm

$\|\vec{v}\|_S^2 = \vec{v}^T S \vec{v}$ S-norm

$\langle v, w \rangle_S = \vec{v}^T S \vec{w}$ S-inner product

# Matrix Norms

## Properties:

① $\|A\| > 0$ if $A \neq 0$ (positive)
$\|A\| = 0$ when $A = 0$

② $\|\alpha A\| = |\alpha| \|A\|$ (Scaling)

③ $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality)

For p-norms and Frobenius norm

④ $\|AB\| \leq \|A\| \|B\|$ (submultiplicative property)

## Frobenius norm

$$\|A\|_F^2 = |a_{11}|^2 + \dots + |a_{1n}|^2 + \dots + |a_{mn}|^2$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2$$

Properties

$Q$ orthogonal

$$\|QB\|_F = \|B\|_F$$

- therefore $\|A\|_F = \|U\Sigma V^T\|_F = \|\Sigma\|_F = (\sigma_1^2 + \dots + \sigma_n^2)^{1/2}$
- $\|A\|_F^2 = \text{trace}(A^T A) = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2$

## §2.3.2 Matrix Norms

**induced matrix norm:**

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{x \neq 0} \left\| A \frac{x}{\|x\|} \right\| = \max_{\|y\|=1} \|Ay\|$$

$$\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}| \quad \text{[max column sum]}$$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \quad \text{[max row sum]}$$

(agree w/ corresponding vector norms for $n \times 1$ matrix)

- **Example**

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix} \Rightarrow \|A\|_1 = 6, \ \|A\|_\infty = 8$$

**Matrix norm properties**

$\Delta$ norm

1. $\|A\| > 0$ if $A \neq 0$
2. $\|\alpha A\| = |\alpha| \, \|A\|$
3. $\|A+B\| \leq \|A\| + \|B\|$

For p-norms

4. $\|AB\| \leq \|A\| \, \|B\|$
5. $\|Ax\| \leq \|A\| \, \|x\|$

"submultiplicative"