Singular Value Decomposition

When A is real, symmetric, we have this nice decomposition

$$A = Q \Lambda Q^T \quad , \quad \Lambda \text{ real}, \; QQ^T = Q^TQ = I$$

But for other matrices, λ complex, eigenvec's not orthogonal.

For non-square A, no eigenvalues/eigenvectors.

Want a more general factorization

Instead of eigenvectors, singular vectors

Need two sets. For A $m \times n$

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^n$ and

$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m \in \mathbb{R}^m$

both u's and v's are orthonormal sets

If A is rank r, we have

$$\left.\begin{aligned} Av_1 &= \sigma_1 u_1 \\ Av_2 &= \sigma_2 u_2 \\ &\vdots \\ Av_r &= \sigma_r u_r \end{aligned}\right\} \text{first } r$$

$$\left.\begin{aligned} Av_{r+1} &= 0 \\ &\vdots \\ Av_n &= 0 \end{aligned}\right\} n-r$$

We order these so that

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

$\vec{v}_{r+1}, \dots, \vec{v}_n$ are the nullspace of A

$\vec{u}_{r+1}, \dots, \vec{u}_m$ are the nullspace of A^T

Write these in matrix form

$$AV = U\Sigma$$

$$A\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{pmatrix}\left(\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ & & & \sigma_r & \\ \hline & & 0 & & 0 \end{array}\right)$$

$v_1, \dots, v_r$ basis for row space of A

$u_1, \dots, u_r$ basis for col space of A

$$VV^T = V^TV = I_{n \times n}$$

$$UU^T = U^TU = I_{m \times m}$$

Singular Value Decomposition $\boxed{A = U\Sigma V^T}$

By multiplying columns of $U\Sigma$ by rows of V^T

$$A = (U\Sigma)V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \dots + \sigma_r u_r v_r^T$$

<u>Example</u>

$$AV = U\Sigma \qquad \overbrace{\begin{pmatrix}3 & 0\\ 4 & 5\end{pmatrix}}^{A} \overbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}}^{V} = \overbrace{\frac{1}{\sqrt{10}}\begin{pmatrix}1 & -3\\ 3 & 1\end{pmatrix}}^{U} \overbrace{\begin{pmatrix}3\sqrt{5} & \\ & \sqrt{5}\end{pmatrix}}^{\Sigma}$$

$$\sigma_1 = 3\sqrt{5}\ , \quad \sigma_2 = \sqrt{5}$$

$$A = U\Sigma V^T$$

$$\begin{pmatrix}3 & 0\\ 4 & 5\end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix}1 & -3\\ 3 & 1\end{pmatrix}\begin{pmatrix}3\sqrt{5} & \\ & \sqrt{5}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$$

$$\begin{pmatrix}3 & 0\\ 4 & 5\end{pmatrix} = 3\sqrt{5}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{10}}\begin{pmatrix}1\\ 3\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} + \sqrt{5}\frac{1}{\sqrt{10}}\frac{1}{\sqrt{2}}\begin{pmatrix}-3\\ 1\end{pmatrix}\begin{pmatrix}-1 & 1\end{pmatrix}$$

$$= \frac{3}{2}\begin{pmatrix}1 & 1\\ 3 & 3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}3 & -3\\ -1 & 1\end{pmatrix} = \begin{pmatrix}3 & 0\\ 4 & 5\end{pmatrix} \checkmark$$

Reduced Form of the SVD

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \dots + \sigma_r u_r v_r^T$$

In matrix form

$$\underset{m \times n}{A} = \underset{m \times r}{\begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_r \\ | & | & & | \end{pmatrix}} \underset{r \times r}{\begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{pmatrix}} \underset{r \times n}{\begin{pmatrix} — & v_1^T & — \\ — & v_2^T & — \\ & \vdots & \\ — & v_r^T & — \end{pmatrix}}$$

Reduced SVD

$$A = U_r \Sigma_r V_r^T$$

Important fact

$$A_k = \sigma_1 u_1 v_1^T + \dots + \sigma_k u_k v_k^T$$

is the best rank k approximation to A.

① Find U, V^T, Σ

by looking at

$A^T A$

$A A^T$

② <u>Example</u> :

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}, \quad U = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$$

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sqrt{45} & \\ & \sqrt{5} \end{pmatrix}$$

③ <u>Polar Decomposition</u>.

$$A = U \Sigma V^T$$

$$= U \Sigma U^T U V^T = S_1 Q$$

or

$$= \cancel{U \Sigma V^T V U \Sigma V^T}$$

$$= U \cancel{\Sigma} V^T V \Sigma V^T$$

$$= Q S_2$$

Find U, Σ, V^T s.t. $A = U\Sigma V^T$

A is $m \times n$

Note:

$$A^TA = V\Sigma^T \cancel{U^T U}\Sigma V^T = V\Sigma^T\Sigma V^T \quad n\times n \quad (1)$$

$$AA^T = U\Sigma \cancel{V^T V}\Sigma^T U^T = U\Sigma\Sigma^T U^T \quad m\times m \quad (2)$$

These take the form of eigen decompositions of symmetric matrices. Specifically,

- the columns of V are the eigenvectors of A^TA
- the columns of U are the eigenvectors of AA^T

To construct $A = U\Sigma V^T$

- Form A^TA and find its eigenvalues and eigenvectors (orthogonal), $\vec{v}_1, \dots, \vec{v}_n$

$$A^TA\, v_k = \lambda_k v_k = \sigma_k^2 v_k \quad (3)$$

That is, $\sigma_k = \sqrt{\lambda_k}$

- Choose $u_1, \dots, u_r$ as follows

$$A v_k = \sigma_k u_k \Rightarrow u_k = \frac{1}{\sigma_k} A v_k$$

To check that the above procedure works, we need to verify that u_k's chosen this way do indeed form an orthonormal set: (i.e., $u_k^T u_j = \delta_{kj} = \begin{cases} 1 & k=j \\ 0 & k\neq j \end{cases}$)

$$u_k^T u_j = \frac{1}{\sigma_k}(Av_k)^T \frac{1}{\sigma_j} A v_j = \frac{1}{\sigma_k\sigma_j} v_k^T A^T A v_j = \frac{1}{\sigma_k\sigma_j} v_k^T \lambda_j v_j \quad \text{by } (3)$$

$$= \frac{\lambda_j}{\sigma_k\sigma_j} v_k^T v_j = \frac{\sigma_j^2}{\sigma_k\sigma_j} v_k^T v_j = \frac{\sigma_j}{\sigma_k}\delta_{jk} = \delta_{jk} \checkmark$$