Strang I. 7 Symmetric Positive Definite Matrices

Symmetric Matrix , $n \times n$

- all n λ's are <u>real</u>
- all eigenvectors are independent and can be made <u>orthogonal</u>

Spectral Theorem

Every real, symmetric matrix has the form $S = Q \Lambda Q^T$

Positive Definite Matrices

A positive definite matrix has all positive eigenvalues "SPD"

Examples

1. $S = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$
2. $S = Q \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} Q^T$, $Q^{-1} = Q^T$
3. $S = C \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} C^T$, C invertible

Examples (cont.)

4. $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ spd when $a > 0$ and $ac - b^2 > 0$

5. $S = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ only positive semi-definite

Energy-based Definition

For S symmetric, S is spd

$$\iff \quad \vec{x}^T S \vec{x} > 0 \quad \forall \vec{x} \neq 0$$

Examples

1. $S = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$

$$x^T S x = (x_1 \ x_2) \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 + 6x_2^2$$

$$> 0 \quad \forall \ (x_1, x_2) \neq (0,0)$$

2. $S = \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix}$

$$x^T S x = (x_1 \ x_2) \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 + 8x_1x_2 + 9x_2^2$$

$$= 2(x_1 + 2x_2)^2 + x_2^2 > 0 \quad \forall \ (x_1, x_2) \neq (0,0)$$

How can we see that this energy test is equivalent to all $\lambda > 0$?

First, consider eigenvector $\vec{x}$ of S,

$$\vec{x}^T(S\vec{x}) = \vec{x}^T\lambda\vec{x} = \lambda x^T x = \lambda \|x\|_2^2$$

$$> 0 \iff \lambda > 0$$

Next, consider arbitrary vector $\vec{v} \neq \vec{0}$.

Since S is real symmetric,

$$S = Q \Lambda Q^T$$

and we can write $\vec{v}$ as

$$\vec{v} = \alpha_1 \vec{q}_1 + \alpha_2 \vec{q}_2 + \dots + \alpha_n \vec{q}_n$$

Then

$$\begin{aligned} v^T S v &= v^T S(\alpha_1 q_1 + \dots + \alpha_n q_n) \\ &= v^T(\alpha_1 \lambda_1 q_1 + \dots + \alpha_n \lambda_n q_n) \\ &= (\alpha_1 q_1^T + \dots + \alpha_n q_n^T)(\alpha_1 \lambda_1 q_1 + \dots + \alpha_n \lambda_n q_n) \end{aligned}$$

<cross terms $q_i^T q_j$ $(i \neq j)$ vanish due to orthogonality>

$$= \alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2 + \dots + \alpha_n^2 \lambda_n$$

$$> 0 \iff \lambda_i > 0 \quad \forall i$$

for all choices of α_k

Other equivalent conditions for S s.p.d :

- $S = A^T A$, A has indep. columns
- leading determinants $D_1, D_2, \dots, D_n > 0$
- all pivots of S positive (in elimination)

<u>$S = A^T A$</u> , two constructions

1. $S = LU = LDL^T$

$$S = (L\sqrt{D})(\sqrt{D}L^T) = A^T A$$

2. $S = Q\Lambda Q^T = Q\sqrt{\Lambda}\sqrt{\Lambda}Q^T$

$$= Q\sqrt{\Lambda}Q^T \, Q\sqrt{\Lambda}Q^T = A^T A = AA$$

$$S = A^T A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \\ 4 & 7 & 10 \end{pmatrix}$$

<u>not</u> spd

E.g. $x = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \implies x^T S x = x^T A^T A x$

$$= x^T A^T \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = x^T A^T \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

S is <u>semi-definite</u> :

$$x^T S x \geq 0$$

Cholesky Factorization

Ex. $S = \begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix}$

LU

$$\begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix}$$

LDL^T

$$\begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Cholesky

$$S = LL^T = l_1 l_1^T + l_2 l_2^T + \dots + l_n l_n^T$$

$$\begin{pmatrix} 2 & 4 \\ 4 & 9 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{2} & 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{2} & 2\sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & 0 \\ 2\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 1 \end{pmatrix}$$

Cholesky Factorization

for $k = 1, \dots, n$

$\quad l_{kk} = \sqrt{a_{kk}}$

$\quad$ for $i = k+1, \dots, n$

$\quad\quad l_{ik} = a_{ik} / l_{kk}$

$\quad$ end

$\quad$ for $i = k+1, \dots, n$

$\quad\quad$ for $j = k+1, \dots, \cancel{n}\ i$ $\qquad$ Save half by symmetry

$\quad\quad\quad a_{ij} \leftarrow a_{ij} - l_{ik} l_{jk}$

$\quad\quad$ end

$\quad$ end

Since we can avoid half of the dominant term, Cholesky costs

$\sim \frac{1}{3} n^3$ operations

half of LU