I.6. Eigenvalues and Eigenvectors

$$A\vec{x} = \lambda\vec{x}$$

$\vec{x}$: eigenvector of A

λ : eigenvalue of A

$$A^2\vec{x} = AA\vec{x} = A\lambda\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x}$$

Generally,

$$A^k\vec{x} = \lambda^k\vec{x}$$

Also, $A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$, $\lambda \neq 0$

Most matrices have n distinct eigenvalues $\lambda_1, \dots, \lambda_n$ with n linearly independent eigenvectorr.

Then every vector $\vec{v} \in \mathbb{R}^n$ can be written

$$\vec{v} = \alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \dots + \alpha_n\vec{x}_n$$

$$A\vec{v} = \alpha_1\lambda_1\vec{x}_1 + \alpha_2\lambda_2\vec{x}_2 + \dots + \alpha_n\lambda_n\vec{x}_n$$

$$A^k\vec{v} = \alpha_1\lambda_1^k\vec{x}_1 + \alpha_2\lambda_2^k\vec{x}_2 + \dots + \alpha_n\lambda_n^k\vec{x}_n$$

Note: eigenvectors can be scaled

$$Ax = \lambda x$$

$$A(\alpha x) = \lambda(\alpha x) \quad \checkmark$$

Convenient to take $\|x\|_2 = 1$

$S_\lambda = \{ x \mid Ax = \lambda x \}$ eigenspace
"invariant subspace"

$\dim S_\lambda$ = geometric multiplicity of λ

Example 1

$$S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$S\vec{x} = \lambda x \Rightarrow (S - \lambda I)\vec{x} = \vec{0}$$

$$\Rightarrow \det(S - \lambda I) = 0 \quad \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0$$

$$\lambda_{1,2} = 1, 3$$

$$S\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 2x + y = x \\ x + 2y = y \end{matrix} \Rightarrow \begin{matrix} x + y = 0 \\ x = -y \end{matrix}$$

$$\lambda_1 = 1, \quad \vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$S\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 2x + y = 3x \\ x + 2y = 3y \end{matrix} \Rightarrow y = x$$

$$\lambda_2 = 3, \quad \vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Powers S^k will grow like 3^k. Note:

- $\text{trace}(S) = 4 = \lambda_1 + \lambda_2$
- $\det(S) = 3 = \lambda_1 \lambda_2$
- λ_1, λ_2 are real.
- orthogonal eigenvectors

Symmetric matrices have real λ. + orthogonal eigenvectors intuition:

symmetric matrices like real numbers

orthogonal matrices like complex numbers with unit length

Example 2

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ rotate $\frac{\pi}{2}$ ccw

$$0 = \det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 \Rightarrow \lambda_{1,2} = \pm i$$

$$Q\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i\begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad Q\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\mathrm{trace}(Q) = 0 = +i - i$ ✓

$\det(Q) = 0 + 1 = (+i)(-i) = -i^2 = 1$ ✓

$\overline{x}_1^T x_2 = \begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = 1 + i^2 = 0$ ✓ orthogonal eigenvectors

Remarks on eigenvalues + eigenvectors

- **not** generally true

$$\lambda(A+B) = \lambda(A) + \lambda(B)$$

$$\lambda(AB) = \lambda(A)\lambda(B)$$

- $\lambda_1 = \lambda_2$ may or may not have two independent eigenvectors

- eigenvectors of real matrix A are orthogonal iff A is normal

$$A^T A = A A^T$$

Example

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

$$A = X \Lambda X^{-1}$$

$$\frac{d\vec{u}}{dt} = X\Lambda X^{-1}\vec{u} \implies \frac{d}{dt} X^{-1}u = \Lambda X^{-1}\vec{u}$$

let $y = X^{-1}u$. $\implies \frac{d}{dt}y = \Lambda y$

$$\implies y(t) = e^{\Lambda t} y(0)$$

$$\implies u(t) = Xy(t) = Xe^{\Lambda t}X^{-1}u(0)$$

$$\vec{u}(0) = X\vec{\alpha}$$

$$\vec{u}(t) = Xe^{\Lambda t}X^{-1}X\vec{\alpha} = Xe^{\Lambda t}\vec{\alpha} = \alpha_1 e^{\lambda_1 t}\vec{x}_1 + \dots + \alpha_n e^{\lambda_n t}\vec{x}_n$$

Finding eigenvalues and eigenvectors

$$Ax = \lambda x \iff (A - \lambda I)x = 0$$

$A - \lambda I$ singular

$\det(A - \lambda I) = 0$

nth degree equation in λ

n roots.

Example $(n = 2)$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= ad - (a+d)\lambda + \lambda^2 - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

Roots (quadratic formula)

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$= \frac{1}{2}\left[(a+d) \pm \sqrt{(a-d)^2 + 4bc}\right]$$

Notes:

- $\lambda_1 + \lambda_2 = a + d = tr(A)$
- A symm $\Rightarrow b = c \Rightarrow \lambda_{1,2} = \frac{1}{2}\left[(a+d) \pm \sqrt{\underbrace{(a-d)^2 + 4b^2}_{\geq 0}}\right]$

  real $\lambda_{1,2}$

Example 3

Find eigvals & eigvecs of

$$A = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & 3 \\ 2 & 7-\lambda \end{vmatrix} = (8-\lambda)(7-\lambda) - 6 = 56 - 15\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 15\lambda + 50 = 0$$

$$\Rightarrow (\lambda - 5)(\lambda - 10) = 0 \qquad \lambda_{1,2} = 5, 10$$

$\lambda_1 = 10$ $\quad \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10x \\ 10y \end{pmatrix} \Rightarrow \cancel{8x} + 3y = \cancel{10x}\ 2x \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix},$

$\lambda_2 = 5$ $\quad \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} \Rightarrow \cancel{8x} + 3y = \cancel{5x}\ {-3x} \quad$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda_1 + \lambda_2 = 10 + 5 = 15$ ✓

$$\vec{X}_1 \cdot \vec{X}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3 - 2 = 1 \neq 0$$

$A = \begin{pmatrix} 8 & 30 \\ 2 & 7 \end{pmatrix}$ has complex λ

Characteristic Polynomial

$$Ax = \lambda x$$

$$\Rightarrow (A - \lambda I)x = 0$$

$$\Leftrightarrow \det(A - \lambda I) = p(\lambda) = 0$$

$p(\lambda)$ is an nth degree polynomial in λ. Its roots are the eigenvalues of A.

The fundamental theorem of algebra tells us that degree n $p(\lambda)$ has n roots.

$$p(\lambda) = a_n(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = 0$$

The roots may be distinct or repeated, real or complex.

A $n \times n$ always has n eigenvalues.

If A is real,

- λ real, or
- λ occur in complex conjugate pairs $\lambda, \bar{\lambda}$

$$(Ax = \lambda x \Rightarrow \bar{A}\bar{x} = \bar{\lambda}\bar{x} \Rightarrow A\bar{x} = \bar{\lambda}\bar{x})$$

But $p(\lambda)$ is not used as a computational method:

- expensive to get coeffs

- coeffs unstable to perturbations in matrix entries

- roots numerically sensitive to forming polynomial

E.g. $A = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$, with $\varepsilon^2 < \varepsilon_{mach}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & \varepsilon \\ \varepsilon & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - \varepsilon^2$$

$$= \lambda^2 - 2\lambda + 1 - \varepsilon^2 \approx \lambda^2 - 2\lambda + 1$$

$$\Longrightarrow \lambda_{1,2} = 1, 1$$

But actual $\lambda_{1,2} = 1+\varepsilon, 1-\varepsilon$

Eigenvalues + Eigenvectors of $A+sI$

"shifted" A

Let $A\vec{x} = \lambda\vec{x}$. Then

$$(A+sI)\vec{x} = A\vec{x} + s\vec{x} = \lambda\vec{x} + s\vec{x}$$
$$= (\lambda + s)\vec{x}$$

So $\lambda + s$ is an eigenvalue of $A+sI$ with eigenvector $\vec{x}$.

Similar Matrices

B invertible

$C = BAB^{-1}$

A and C are <u>similar</u>

Let $A\vec{x} = \lambda\vec{x}$. Let $\vec{y} = B\vec{x}$

Then
$$C\vec{y} = BAB^{-1}\vec{y}$$
$$= BAx = B\lambda\vec{x} = \lambda\vec{y}$$

So $\vec{y} = B\vec{x}$ is eigenvector of $C = BAB^{-1}$ with eigenvalue λ.

A and BAB^{-1} are <u>similar</u>:
same eigenvalues.

use this to compute λ's of large matrices. Gradually make

$$BAB^{-1}$$

triangular. Eigenvalues not changing and showing up on main diagonal.

Eigenvalues of

$$A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

are $\lambda_1 = a$, $\lambda_2 = c$

(note $A - aI$ and $A - cI$ will have $\det = 0$).

Spectral Radius of A

$$\rho(A) = \max \{ |\lambda| : \lambda \in \lambda(A) \}$$

Diagonalizing a Matrix

If A has a full set of eigenvectors

$\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n,$

Let $X = \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{pmatrix}$

Then $AX = \begin{pmatrix} | & | & & | \\ Ax_1 & Ax_2 & \cdots & Ax_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \\ | & | & & | \end{pmatrix}$

$= \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = X\Lambda$

Therefore $\boxed{A = X\Lambda X^{-1}}$

Λ = diagonal eigenvalue matrix

X = invertible eigenvector matrix.

(Example 3)

$A = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 10 & \\ & 5 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$

$A^k = X\Lambda^k X^{-1}$

To compute $A^k v$:
1. $X^{-1}v$ [coeffs]
2. $\Lambda^k (X^{-1}v)$ [λ^k]
3. $X\Lambda^k X^{-1} v$ [sum]

Nondiagonalizable Matrices

Geometric: $A\vec{x} = \lambda \vec{x}$

Algebraic: $\det(A - \lambda I) = 0$

(GM) **geometric multiplicity** of λ

# of indep. eigenvec associated with λ

(AM) **algebraic multiplicity** of λ

# of repetitions of λ among eigenvalues

roots of $\det(A - \lambda I) = 0$

$$(GM) \leq (AM)$$

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \lambda_{1,2} = 0$, but 1 eigenvector

$AM(0) = 2$ $\quad GM < AM \Rightarrow$

$GM(0) = 1$

A is <u>not</u> diagonalizable

<u>additional examples</u>: $\lambda_{1,2} = 5$ $\quad AM = 2, \; GM = 1$

$$A = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}, \quad A = \begin{pmatrix} 6 & -1 \\ 1 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 7 & 2 \\ -2 & 3 \end{pmatrix}$$

"**defective**"

$\text{rank}(A - 5I) = 1$

nullspace $(A - 5I)$ has dim 1

Symmetric Matrix

$A = A^T$, A real

Can write

$$\boxed{A = Q \Lambda Q^T}$$

Λ are real

Q is an orthogonal matrix $(QQ^T = Q^T Q = I)$

Hermitian Matrix

$A = A^H$, A complex

Can write

$$\boxed{A = Q \Lambda Q^H}$$

Λ are real

Q is a unitary matrix $(QQ^H = Q^H Q = I)$

Normal Matrix

- matrices that are <u>unitarily diagonalizable</u>

- special cases: Hermitian, unitary, and skew-Hermitian matrices

<u>definition</u>: A is normal iff

$$AA^* - A^*A = 0$$

<u>Example</u>:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$AA^T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \checkmark$$

$$\parallel$$

$$A^TA = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$A = Q\Lambda Q^H$ $\qquad$ Λ real $\Rightarrow$ A Hermitian

Λ complex $\Rightarrow$ A Normal