Range and Nullspace $\quad A \in \mathbb{R}^{m \times n}$

range = columnspace $= \{ A\vec{x} \mid \forall \vec{x} \in \mathbb{R}^n \}$

nullspace $= \{ \vec{z} \mid A\vec{z} = \vec{0} \}$

rank = dim (col space)

Four fundamental subspaces (Fig. I.3)

$\mathbb{R}^n$		$\mathbb{R}^m$	
$C(A^T)$	dim r	$C(A)$	dim r
$N(A)$	dim $n-r$	$N(A^T)$	dim $m-r$
$C(A^T) \perp N(A)$		$C(A) \perp N(A^T)$	

I.3 The Four Fundamental Subspaces

$A \in \mathbb{R}^{m \times n}$

two subspaces of $\mathbb{R}^n$

two subspaces of $\mathbb{R}^m$

Examples: rank 1 matrix, 2×3 matrix,
5×4 incidence matrix of graph

$\boxed{\textbf{Ex. 1}}$ $\quad A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}$

1. column space $C(A)$: line through $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$
2. row space $C(A^T)$: line through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$
3. nullspace $N(A)$: line through $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\quad Ax = 0$
4. left nullspace $N(A^T)$: line through $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\quad A^T y = 0$

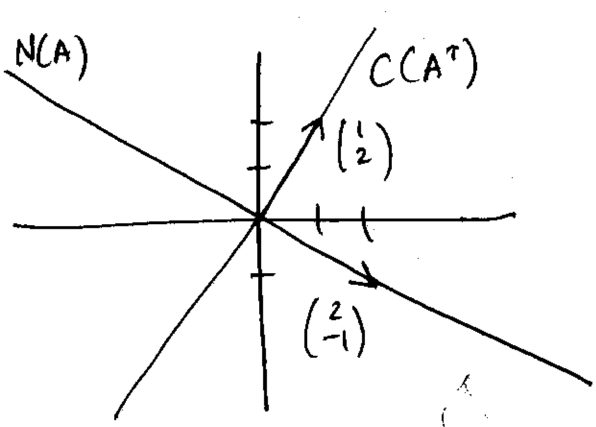
Definitions:

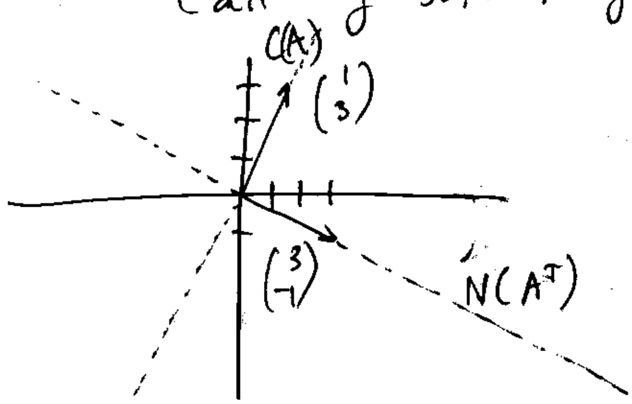
columnspace $C(A)$ = all combinations of cols. of A

row space $C(A^T)$ = all combinations of rows. of A
(all combinations of cols of A^T)

nullspace $N(A)$ = all x s.t. $Ax = 0$

left nullspace $N(A^T)$ = all y s.t. $y^T A = 0$
(all y s.t. $A^T y = 0$)





$\boxed{\text{Ex. 2}}$ $B = \begin{pmatrix} 1 & -2 & -2 \\ 3 & -6 & -6 \end{pmatrix}$, $m=2$, $n=3$

$C(B)$: line through $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$C(B^T)$: line through $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

To find nullspace(B),

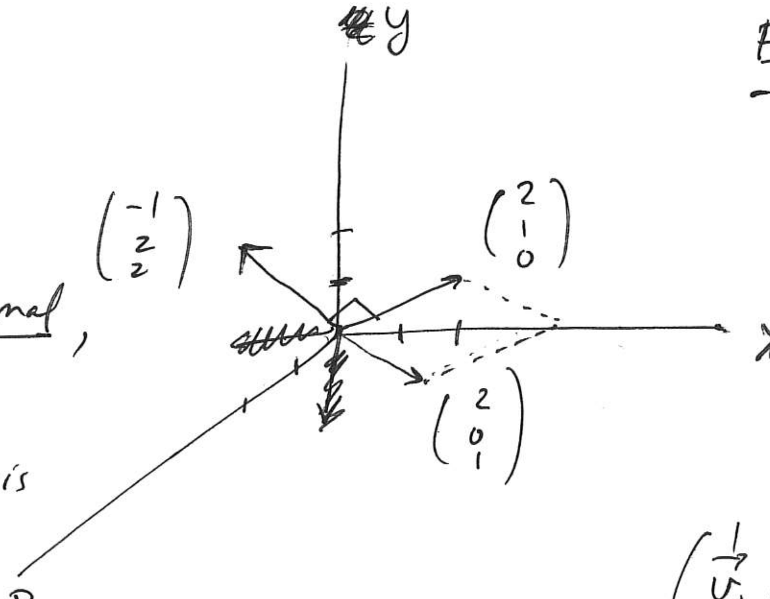
$$B\vec{x} = \begin{pmatrix} 1 & -2 & -2 \\ 3 & -6 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

note $\underline{1}$ indep. eq. and $\underline{3}$ unknowns $\Rightarrow \dim(N(B)) = 3-1 = 2$

$\vec{x}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, and $\vec{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ are indep. solutions.

note, however $\vec{x}_1$ & $\vec{x}_2$ are not perpendicular ($\vec{x}_1 \cdot \vec{x}_2 = 4 \neq 0$). We will prefer perpendicular basis vecs. ~~&~~ Can be generated from $\vec{x}_1, \vec{x}_2$ by Gram-Schmidt.

$N(B)$ = plane in $\mathbb{R}^3$



this system can also be made $\underline{\text{orthonormal}}$, giving an orthonormal basis for $\mathbb{R}^3$

$\underline{\text{E.g.}}$,

$\vec{v}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

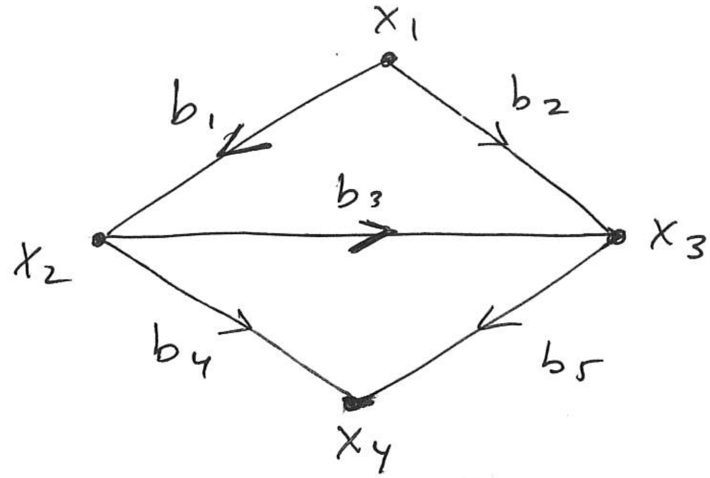
$\vec{v}_2 = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$\vec{v}_3 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$ orthonormal basis for $\mathbb{R}^3$

r indep eqs. have $n-r$ indep. solutions $Ax=0$

$\boxed{\text{Ex. 3}}$ graph



1 equation per edge
1 unknown per node

$$
\begin{aligned}
-x_1 + x_2 &= b_1 \\
-x_1 + x_3 &= b_2 \\
-x_2 + x_3 &= b_3 \\
-x_2 + x_4 &= b_4 \\
-x_3 + x_4 &= b_5
\end{aligned}
$$

matrix form

$$
\underbrace{\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}}_{A}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}
=
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}
$$

A = incidence matrix of the graph.

nullspace (A)

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\underset{\substack{\uparrow \\ \text{Eq.2}}}{x_3} = x_1 \underset{\substack{\uparrow \\ \text{Eq.1}}}{=} x_2 \underset{\substack{\uparrow \\ \text{Eq.4}}}{=} x_4 \qquad \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{solution}$$

$\dim(N(A)) = 1$

$\text{rank}(A) = 3 \qquad n - r = 4 - 3 = 1$

Dimensions of all four subspaces:

$\dim C(A^T) = \cancel{3}\; r = 3 \qquad \dim C(A) = \cancel{3}\; r = 3$

$\dim N(A) = n - r = 1 \qquad \dim N(A^T) = m - r = 2$

column space (A) $\quad r = 4 - 1 = 3$

columns 1, 2, 3 of A are independent

column 4 is a combination of those.

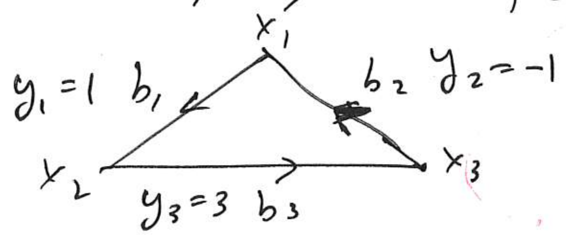
row space $C(A^T)$ $\quad r = 3$, same as for columns

row 3 = row 2 − row 1

rows 1, 2, 4 independent → basis for rowspace.

recall, each row corresponded to an edge.

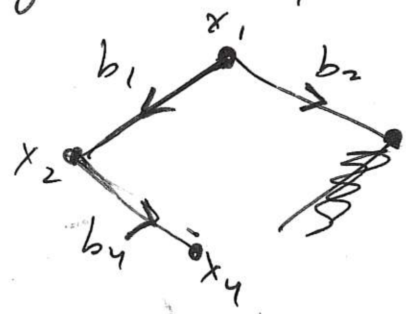
edges ~~loops~~ 1, 2, 3 form a **loop**



$\Rightarrow A^T y = 0$

(dependent rows)

edges 1, 2, 4 form a **tree**



(independent rows)

left nullspace $N(A^T)$

$$A^T y = 0$$

$$y_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

upper loop $\qquad$ lower loop.

$\vec{y}_1, \vec{y}_2$ independent

$$\dim N(A^T) = m - r = 5 - 3 = 2$$

so $\vec{y}_1$ & $\vec{y}_2$ form a basis for $N(A^T)$.

(*) Connected graph with m edges and n nodes, incidence matrix A

Four subspaces:

$N(A)$ constant vector $c(1, 1, \dots, 1)$ make up the 1 dimensional nullspace of A.

$C(A^T)$ r edges of a tree (MST) give r independent rows of A.

rank $= r = n-1$

$C(A)$ $A\vec{x}$ components of $A\vec{x}$ sum to 0 around any closed loop.

$N(A^T)$ $\vec{y}$ s.t. $A^T\vec{y} = \vec{0}$ $\therefore$ solutions are loops.

$m-r$ independent small loops in the graph

$C(A)$ dim r

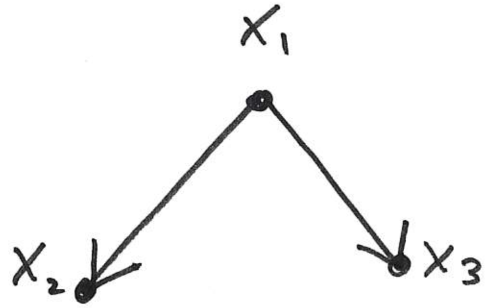
$N(A^T)$ dim $m-r$

dim m

$C(A^T)$ dim r

$N(A)$ dim $n-r$

dim n.



$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

no loops

rows are linearly indep.

$N(A)$ $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ dim = 1

$C(A^T)$ $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ dim = 2 = r

rowspace

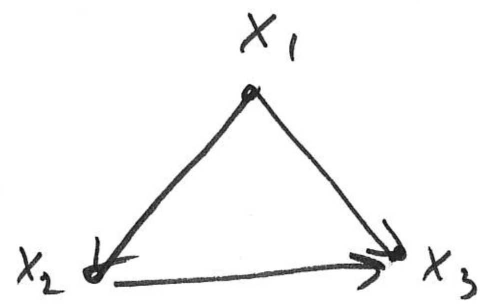
$C(A)$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ dim = 2 = r

columnspace

$N(A^T)$ $\{\vec{0}\}$ dim = 0

$$\underline{\begin{matrix} C(A) & \text{dim} = r \\ N(A^T) & \text{dim} = m-r \end{matrix}}$$
m

$$\underline{\begin{matrix} C(A^T) & \text{dim} = r \\ N(A) & \text{dim} = n-r \end{matrix}}$$
n



$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

loop

3rd row can be written as lin. combin. of 1st + 2nd rows.

$N(A) = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ dim = 1

$C(A^T)$ $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ dim = 2 = r

rowspace

$C(A)$ $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ dim = 2 = r

$N(A^T)$ $(-1, 1, 1)$ dim = 1 (loop)

$$\underline{\begin{matrix} C(A) & \text{dim} = r \\ N(A^T) & \text{dim} = m-r \end{matrix}}$$
m

$$\underline{\begin{matrix} C(A^T) & \text{dim} = r \\ N(A) & \text{dim} = n-r \end{matrix}}$$
n

Ranks of AB and $A+B$

When we multiply matrices, the **ranks cannot increase**.

1. $\text{rank}(AB) \le \text{rank}(A)$

   $\text{rank}(AB) \le \text{rank}(B)$

2. $\text{rank}(A+B) \le \text{rank}(A) + \text{rank}(B)$

3. $\text{rank}(A^TA) = \text{rank}(AA^T) = \text{rank}(A)$

   $= \text{rank}(A^T)$

4. $A \in \mathbb{R}^{m\times r}$, $B \in \mathbb{R}^{r\times n}$, $\text{rank}(A) = \text{rank}(B) = r$

   $AB \in \mathbb{R}^{m\times n}$ $\quad \text{rank}(AB) = r$

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1. $C(AB) \subseteq C(A)$ and $C((AB)^T) \subseteq C(B^T)$

2. $A+B$ each col is sum of col of A and col of B

   combines bases for $C(A)$ and $C(B)$

   $\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)$ not generally true

   e.g. $A = B = I$

3. A and A^TA both have n columns, and same nullspace. dim nullspace is $n-r$, so $r = \text{rank}$ must be same for both.

   $\text{rank}(A^T) \ge \text{rank}(A^TA) = \text{rank}(A)$

   $\text{rank}(A) \ge \text{rank}(AA^T) = \text{rank}(A^T)$ $\quad \Rightarrow \text{rank}(A) = \text{rank}(A^T)$

4. $A^TA \in \mathbb{R}^{r\times r}$, $BB^T \in \mathbb{R}^{r\times n}$ invertible $\quad r = \text{rank}(A^TABB^T) \le \text{rank}(AB)$

   $\le \text{rank}(A) = r$

   $\Rightarrow \text{rank}(AB) = r$.

Note:

$A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 & -3 \end{pmatrix}$

3×1 1×3

$\text{rank}(A) = 1$ $\text{rank}(B) = 1$

$\text{rank}(AB) = 1$ ✓

$\text{rank}(BA) = 0 \neq 1$

$\underset{m \times r}{A} \ \underset{r \times n}{B}$ has rank r

$\underset{r \times m}{B} \ \underset{m \times r}{A}$ does not necessarily.