Strang I.1 Multiplication Ax using columns of A

Matrix-vector multiplication:

$\boxed{\text{Ex. 1}}$ **by rows**

$$\begin{pmatrix} \boxed{2 \quad 3} \\ 2 \quad 4 \\ 3 \quad 7 \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} x_1 \\ x_2 \end{matrix}} \end{pmatrix} = \begin{pmatrix} \boxed{2x_1 + 3x_2} \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{pmatrix}$$

inner products of rows with $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{x}$

by columns

$$\begin{pmatrix} \boxed{\begin{matrix} 2 \\ 2 \\ 3 \end{matrix}} & \boxed{\begin{matrix} 3 \\ 4 \\ 7 \end{matrix}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ \vec{a}_1 & \vec{a}_2 \end{matrix}$$

combination of the columns $\vec{a}_1$ and $\vec{a}_2$

$\Longrightarrow$ $A\vec{x}$ is a linear combination of the columns of A.

Column Space of A

all linear combinations of the columns of A

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = A\vec{x}$$

Note: $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^3$ Q. what is $x_1 \vec{a}_1 + x_2 \vec{a}_2 \ \forall \vec{x}$?

A. Plane containing the two lines $x_1\vec{a}_1$ and $x_2\vec{a}_2$

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$\vec{b}$ is in the column space of A, $C(A)$, exactly when $A\vec{x} = \vec{b}$ has a solution (x_1, x_2)

$\boxed{\text{Ex. 2}}$ $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ <u>not</u> in $C(A)$

$$Ax = \begin{pmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$ unsolvable

$$\begin{array}{l} 2x_1 + 3x_2 = 1 \\ -(2x_1 + 4x_2 = 1) \\ \hline 0 - x_2 = 0 \end{array} \Rightarrow \boxed{x_2 = 0}$$

$2x_1 + 3 \cdot 0 = 1 \Rightarrow \boxed{x_1 = \frac{1}{2}}$

But $3 \cdot \frac{1}{2} + 7 \cdot 0 = \frac{3}{2} \neq 1$

$\boxed{\text{Ex. 3}}$ What are the column spaces of

$$A_2 = \begin{pmatrix} 2 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 7 & 10 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 1 \end{pmatrix} ?$$

$C(A_2) = C(A)$ and $C(A_3) = \mathbb{R}^3$

All possible column spaces inside $\mathbb{R}^3$:

Subspaces of $\mathbb{R}^3$:

zero vec	$\vec{0} = (0,0,0)$	dim = 0
line	$x_1\vec{a}_1$	dim = 1
plane	$x_1\vec{a}_1 + x_2\vec{a}_2$	dim = 2
$\mathbb{R}^3$	$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$	dim = 3

($\vec{a}_1, \vec{a}_2, \vec{a}_3$ are independent)

note: $\vec{0}$ is in every subspace

- three independant columns in $\mathbb{R}^3$ give an an invertible matrix $AA^{-1} = A^{-1}A = I$

- $Ax = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

  Then $Ax = b$ has exactly one solution $x = A^{-1}b$

Columns of $n \times n$ invertible matrix are lineally independent. Their combinations fill all of $\mathbb{R}^n$.

Independent Columns and Rank of A

(basis = full set of linearly indep. vecs.)

Let's construct a **basis** for columnspace of A out of the columns of A.

~~[illegible]~~ and put these basis vectors into a matrix C, so that we can write

$$A = CR$$

Idea:

If $\vec{a}_1 \neq \vec{0}$, put $\vec{a}_1$ into C

If $\vec{a}_2 \neq \alpha \vec{a}_1$, put $\vec{a}_2$ into C

If $\vec{a}_3 \neq \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2$, put $\vec{a}_3$ into C

$\vdots$

then C will have $r \leq n$ columns

$\boxed{\text{Ex. 4}}$

$$A = \begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix} \Rightarrow C = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$n = 3$ cols in A $\qquad$ $r = 2$ cols in C

$\boxed{\text{Ex. 5}}$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \Rightarrow C = A$$

$n = 3$ cols in A $\qquad$ $r = 3$ cols in C

Ex. 6

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{pmatrix} \Rightarrow C = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$n = 3$ cols in A $\qquad$ $r = 1$ col in C

r is the **rank of A**

(and the rank of C)

Note, we could start right & go left in cols of A. Change basis but not # of indep. vecs.

r is dimension of columnspace of A.

The <u>rank</u> of a matrix is the dimension of its column space.

Fill in R $\qquad A = CR$

$\boxed{\text{Ex. 4}}$
$$\begin{pmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \qquad \text{rank} = 2$$
$$3\times 3 \qquad 3\times 2 \qquad 2\times 3$$

$\boxed{\text{Ex. 5}}$
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{rank} = 3$$
$$3\times 3 \qquad 3\times 3 \qquad 3\times 3$$

$\boxed{\text{Ex. 6}}$
$$\begin{pmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5 \end{pmatrix} \qquad \text{rank} = 1$$
$$3\times 3 \qquad 3\times 1 \qquad 1\times 3$$

Note R is the reduced row-echelon form of A.

All 3 matrices in ~~all~~ each ~~three~~ examples have the same rank (A, C, R)

indep cols = # indep rows

- R has r rows
- rows of R form basis for row space of A

and

- rows of R are linearly independent

(because in each row the leading one is to the right of the previous rows)

$\Rightarrow$ dim of rowspace of A is also r.

When we compute SVD,

$$A = C R$$

look for C w/ orthogonal cols

and R w/ " rows.

I.2 Matrix-Matrix Multiplication AB

Usually, to compute $C = AB$, inner products

E.g., $A, B \in \mathbb{R}^{3\times 3}$

$$
\begin{aligned}
C_{23} &= (\text{row 2 of } A) \cdot (\text{col 3 of } B) \\
&= \sum_{k=1}^{3} a_{2k} b_{k3} \\
&= a_{21} b_{13} + a_{22} b_{23} + a_{23} b_{33}
\end{aligned}
$$

$$
\begin{pmatrix} \cdot & \cdot & \cdot \\ \boxed{a_{21} \quad a_{22} \quad a_{23}} \\ \cdot & \cdot & \cdot \end{pmatrix}
\begin{pmatrix} \cdot & \cdot & b_{13} \\ \cdot & \cdot & b_{23} \\ \cdot & \cdot & b_{33} \end{pmatrix} = C_{23}
$$

Another way is columns of A times rows of B.

Outer product matrix rank 1 matrix

$$
\vec{u}\vec{v}^T = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 8 & 12 \\ 6 & 8 & 12 \\ 3 & 4 & 6 \end{pmatrix}
$$

All columns of $\vec{u}\vec{v}^T$ are multiples of $\vec{u}$

All rows of $\vec{u}\vec{v}^T$ are multiples of $\vec{v}^T$

all nonzero $\vec{u}\vec{v}^T$ are rank 1 matrices. Building blocks of matrices.

AB = sum of rank one matrices

$$\underset{m\times n}{A}\;\underset{n\times p}{B} =$$

$$= \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} — & b_1^* & — \\ — & b_2^* & — \\ & \vdots & \\ — & b_n^* & — \end{pmatrix}$$

$$= a_1 b_1^* + a_2 b_2^* + \cdots + a_n b_n^*$$

sum of rank 1 matrices

$\boxed{\text{Example}}$

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}(2\ 4) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0\ 5)$$

$$= \begin{pmatrix} 2 & 4 \\ 6 & 12 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 17 \end{pmatrix}$$

number of multiplications nmp
(n rank 1 matrices of size $m \times p$)

Same # of mults for inner product way of multiplying AB
(mp numbers in AB, each requiring n mults $\Rightarrow mpn$)

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad = \text{row } i \cdot \text{col } j$$

vs. $\quad C = \sum_{k=1}^{n} \vec{a}_k \vec{b}_k^* \Rightarrow c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Insight from Column times Row

We will study several **factorizations** of A

$$A = LU$$

$$A = QR$$

$$A = S = Q\Lambda Q^T \qquad S \text{ symmetric}$$

$$A = X\Lambda X^{-1} \qquad A \text{ nondefective}$$

$$A = U\Sigma V^T$$

last is **SVD**

$$\underset{m\times n}{A} = \underset{m\times m}{U}\ \underset{m\times n}{\Sigma}\ \underset{n\times n}{V^T}, \quad \Sigma \text{ diagonal, positive entries or } 0$$

$$= \sum_{k=1}^{\min(m,n)} \sigma_k u_k v_k^T = \sum_{k=1}^{r} \sigma_k u_k v_k^T, \quad r = \operatorname{rank}(A)$$

sum of rank 1 matrices