CS 210
Final (Practice Problems)

Fall 2019

Name	
Student ID	
Signature	

You may not ask any questions during the exam. If you believe that there is something wrong with a question, write down what you think the question is trying to ask, and answer that.

Question	Points	Score
1	2	
2	2	
3	2	
4	2	
5	2	
6	2	
7	2	
8	4	
9	4	
10	4	
11	4	
12	4	
13	4	
14	4	
15	4	
16	12	
17	12	
18	15	
Total	85	

True/False

For each question, indicate whether the statement is true or false by circling T or F, respectively.

1. (T/F) The normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ for the least squares problem $\min_x \|b - A\mathbf{x}\|_2$ may have no solution.
2. (T/F) The QR decomposition can be stably computed using Householder reflections.
3. (T/F) A Householder matrix is an orthogonal matrix with a determinant of -1.
4. (T/F) Newton's method is an example of a fixed point iteration algorithm.
5. (T/F) If $A = U\Sigma V^T$ is the singular value decomposition of A, then $A^T A = V(\Sigma^T\Sigma)V^T$ is an eigendecomposition of $A^T A$.
6. (T/F) The eigenvalues of a real matrix are real.
7. (T/F) The Steepest Descent Method for unconstrained minimization breaks the problem down into a sequence of 1D minimizations.

Multiple Choice

Instructions: For the multiple choice problems, circle exactly one of (a) - (e).

8. Which of the following statements about the Least Squares (LS) problem $\min_{\mathbf{x}} \|\mathbf{b} - A\mathbf{x}\|_2$ are true?

   I. A solution of the LS problem satisfies $A^T(\mathbf{b} - A\mathbf{x}) = \mathbf{0}$.

   II. If A is rank-deficient, the least squares problem has no solution.

   III. $\|\mathbf{b} - A\mathbf{x}\|_2 = \|M(\mathbf{b} - A\mathbf{x})\|_2$, where M is an elimination matrix as used in LU factorization.

   (a) I only
   (b) II only
   (c) III only
   (d) I and II only
   (e) II and III only

9. Which of the following statements about the Least Squares (LS) problem $\min_{\mathbf{x}} \|\mathbf{b} - A\mathbf{x}\|_2$ are true?

   I. The solution of the LS problem always exists.

   II. The solution of the LS problem is unique if A is full rank.

   III. A solution to the LS problem is $\mathbf{x} = A^+\mathbf{b}$, where A^+ is the pseudoinverse of A.

   (a) I only
   (b) III only
   (c) I and II only
   (d) I and III only
   (e) I, II and III

10. Which of the following statements are true?

   I. An eigenvalue is said to be defective if its algebraic multiplicity is less than its geometric multiplicity.

   II. The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is defective.

   III. Symmetric, real matrices are always nondefective.

   (a) I only
   (b) II only
   (c) III only
   (d) II and III only
   (e) I, II and III

11. Let $A \in \mathbb{R}^{n \times n}$ be invertible. Which of the following matrices have the same eigenvectors as A?

   I. $A - \alpha I$, for some nonzero scalar α.

   II. SAS^{-1}, for some invertible matrix S.

   III. A^{-1}.

   (a) I only
   (b) II only
   (c) III only
   (d) I and II only
   (e) I and III only

12. Consider solving $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$. Let $J_g(\mathbf{x}) = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x})$ be the the Jacobian matrix of $\mathbf{g}$ so that $\mathbf{g}(\mathbf{x}+\mathbf{s}) = \mathbf{g}(\mathbf{x}) + J_g(\mathbf{x})\mathbf{s} + O(\|\mathbf{s}\|^2)$ is the Taylor expansion of $\mathbf{g}$ about $\mathbf{x}$. Which of the following statements are true?

   I. If $\mathbf{g}$ is a quadratic function, then $J_g(\mathbf{x})$ is a constant matrix.

   II. Applying Newton's method, $\mathbf{s}_k = -J_g^{-1}(\mathbf{x}_k)\mathbf{g}(\mathbf{x}_k)$ is the Newton step such that $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{s}_k$.

   III. $J_g(\mathbf{x}) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \frac{\partial g_2}{\partial x_n} & \cdots & \frac{\partial g_n}{\partial x_n} \end{pmatrix}$

   (a) I only
   (b) II only
   (c) III only
   (d) I and III only
   (e) II and III only

13. Consider an unconstrained minimization problem where we are seeking a minimizer $\mathbf{x}^*$ of a function $f(\mathbf{x})$. Which of the following statements are true?

    I. The gradient of f, $\nabla f(\mathbf{x})$, points in a "downhill" direction of f.

    II. A critical point $\mathbf{x}^*$ of f is a minimizer of f if the Hessian matrix $H_f(\mathbf{x}^*)$ is negative definite.

    III. A necessary condition for f to have a minimum at $\mathbf{x}^*$ is that $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

    (a) I only
    (b) II only
    (c) III only
    (d) I and III only
    (e) II and III only

14. Consider an unconstrained minimization problem where we are seeking a minimizer $\mathbf{x}^*$ of a function $f(\mathbf{x})$. Which of the following statements about line search methods are true?

    I. Exact line search methods look for a local minimum of the function along the search direction, while inexact line search methods attempt to make sufficient progress in reducing the function.

    II. Considering a step size α_k along a search direction $\mathbf{s}_k$, the Armijo condition is satisfied whenever $f(x_k + \alpha_k s_k) < f(x_k)$.

    III. A descent direction at $\mathbf{x}^k$ is a direction such that $\nabla f(\mathbf{x}_k)^T \mathbf{s}_k < 0$.

    (a) I only
    (b) II only
    (c) III only
    (d) I and III only
    (e) II and III only

15. Let $A \in \mathbb{R}^{n \times n}$ be invertible. Consider solving $A\mathbf{x} = \mathbf{b}$ through the fixed point iteration $M\mathbf{x}^{k+1} = N\mathbf{x}^k + \mathbf{b}$, where $A = M - N$. Which statements are true?

    I. The iteration is convergent if $\rho(M^{-1}N) < 1$, where $\rho(A)$ is the spectral radius of A.

    II. Gauss-Seidel iterations involve solution of a diagonal system, while Jacobi iterations involve solution of a triangular system.

    III. The iteration $D\mathbf{x}^{k+1} = -(U + L)\mathbf{x}^k + b$, where D contains the diagonal elements of A and $U + L$ contain the off-diagonal elements, can be updated in parallel for each component of $\mathbf{x}^{k+1}$.

    (a) I only
    (b) II only
    (c) III only
    (d) I and II only
    (e) I and III only

Written Response

16. *Nonlinear Equations: Newton's Method.* Consider the nonlinear equation

$$f(x) = 0,$$

for $f : \mathbb{R} \to \mathbb{R}$.

(a) Write down Newton's method for solving this equation.

(b) Let $f(x) = x^2 - 4$. Starting with $x_0 = 3$, carry out one step of Newton's method.

(c) What will be the convergence rate of Newton's method in this case?

(d) Let $f(x) = x^2$. What will be the convergence rate of Newton's method?

17. *Optimization.* Consider the function

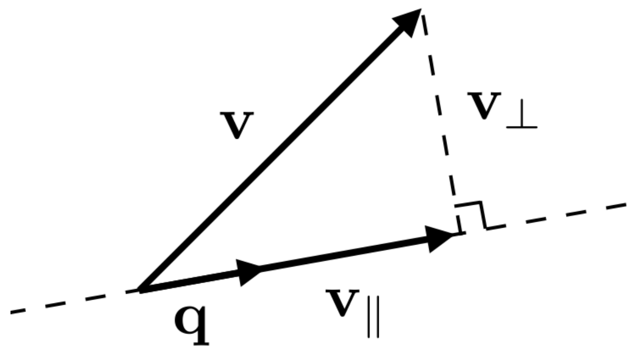
$$f(x,y) = 4x^2 + 2y^2 + 2xy - 2x - 4y + 1.$$

(a) Find $\nabla f(x,y)$.

(b) Find the stationary points of f.

(c) Find the Hessian $H_f(x,y)$ of f.

(d) Classify the stationary points of f as minima, maxima, or saddle points.



18. *Orthgonalization.* Let $\mathbf{q}, \mathbf{v} \in \mathbb{R}^n$ and let $\mathbf{q}$ be a unit vector, i.e. $\|\mathbf{q}\|_2 = 1$, as illustrated in the figure above.

    (a) Give an expression for $\mathbf{v}_{\|}$, the projection of $\mathbf{v}$ onto the direction $\mathbf{q}$.

    (b) Give an expression for $\mathbf{v}_{\perp}$, the component of $\mathbf{v}$ orthogonal to $\mathbf{q}$.

    (c) Write down the projector matrix onto range of $\mathbf{q}$. What is the rank of this matrix?

    (d) Write down the complementary projector to the projector in part (c). What is the rank of this matrix?

    (e) Find the eigenvalues and eigenvectors of the matrix in part (c).