# CS 210
# Midterm

Fall 2019

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| Name | |
| Student ID | |
| Signature | |

You may not ask any questions during the exam. If you believe that there is something wrong with a question, write down what you think the question is trying to ask, and answer that.

| Question | Points | Score |
|---|---|---|
| 1. | 2 | |
| 2. | 2 | |
| 3. | 2 | |
| 4. | 2 | |
| 5. | 2 | |
| 6. | 2 | |
| 7. | 4 | |
| 8. | 4 | |
| 9. | 4 | |
| 10. | 4 | |
| 11. | 4 | |
| 12. | 16 | |
| 13. | 12 | |
| Total | 60 | |

## True/False

For each question, indicate whether the statement is true or false by circling T or F, respectively.

1. (T/F) Examples of unstable algorithms include Gaussian Elimination without pivoting and classical Gram-Schmidt orthogonalization.

2. (T/F) If $A$ is singular, then $A\mathbf{x} = \mathbf{b}$ will have infinitely many solutions.

3. (T/F) Solving $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{n\times n}$ a diagonal matrix, requires $\sim n^2$ operations.

4. (T/F) Cholesky factorization of a symmetric, positive definite matrix requires pivoting to be stable.

5. (T/F) An invertible square matrix $A$ does not have any 0 singular values, but can have both positive and negative singular values.

6. (T/F) A Householder matrix is an orthogonal matrix with a determinant of $-1$.

## Multiple Choice

Instructions: For the multiple choice problems, circle exactly one of (a) - (e).

7. Which one of the following statements is false?

   (a) $\|A\|_2 = \sigma_1$, where $\sigma_1$ is the largest singular value of a real matrix $A$.

   (b) $\|A^{-1}\|_2 = \frac{1}{\sigma_1}$, where $\sigma_1$ is the largest singular value of an invertible, real matrix $A$.

   (c) If $\|\cdot\|_q$ and $\|\cdot\|_p$ are both vector p-norms, then they are equivalent, i.e., there exist constants $C_1$ and $C_2$ such that $C_1\|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p \leq C_2\|\mathbf{x}\|_q$ for all vectors $\mathbf{x}$.

   (d) An orthogonal matrix, $Q$, satisfies $\|Q\|_2 = 1$.

   (e) For any vector $\mathbf{x}$, $\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2$.

8. Let $A$ be an $n \times n$ matrix. Which of the following properties would necessarily imply that $A$ is singular?

   I. The rows of $A$ are linearly dependent.

   II. A zero diagonal element is encountered while performing LU factorization (without pivoting).

   III. $A\mathbf{z} = \mathbf{0}$, for some $\mathbf{z} \neq \mathbf{0}$.

   (a) II only

   (b) I and II only

   (c) I and III only

   (d) II and III only

   (e) I, II and III

9. Which of the following statements is false?

   (a) The number of solutions of $A\mathbf{x} = \mathbf{b}$ may depend on $\mathbf{b}$.

   (b) If $A$ is singular, then $A\mathbf{x} = \mathbf{b}$ has either no solution or infinitely many solutions.

   (c) If $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$, then $\mathbf{b}$ must be in the columnspace of $A$.

   (d) Solving a triangular system by forward or backward substitution requires $O(n^2)$ flops.

   (e) The LU and Cholesky factorizations of a symmetric, positive definite matrix would be exactly the same.

10. Which of the following statements about the Singular Value Decomposition (SVD) are true?

    I. Every real matrix has an SVD.

    II. If a matrix Q is orthogonal, then its singular values are all 1.

    III. A matrix with rank $r$ will have exactly $r$ singular values that are greater than 0.

    (a) I only
    (b) I and II only
    (c) I and III only
    (d) II and III only
    (e) I, II and III

11. Let $A = U\Sigma V^T$ be the Singular Value Decomposition (SVD) of the matrix $A \in \mathcal{R}^{m\times n}$ and let $A^+$ denote the pseudoinverse of $A$. Which of the following statements are true?

    I. $\text{rank}(A) \leq \min(m, n)$.

    II. $A^+ = V\Sigma^+U^T$ where $\Sigma^+$ is the pseudoinverse of $\Sigma$.

    III. The columns of $U$ are mutually orthogonal, but need not be of unit length.

    (a) I only
    (b) II only
    (c) III only
    (d) I and II only
    (e) I and III only

## Written Response

12. *Linear Systems and LU Factorization.*

   (a) Given a nonsingular system of linear equations $A\mathbf{x} = \mathbf{b}$, what effect on the solution vector $\mathbf{x}$ results from each of the following actions?

      i. Permuting the rows of $[A \quad \mathbf{b}]$.

      ii. Permuting the columns of $A$.

      iii. Multiplying both sides of the equation from the left by a nonsingular matrix $M$.

   (b) Consider the symmetric matrix

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 4 & 11 & 15 \\ 6 & 15 & 23 \end{pmatrix}.$$

   - Find unit lower triangular matrix $L$ and upper triangular matrix $U$ such that $A = LU$.
   - Use the factorization $A = LU$ that you found above to express $A$ as $A = LDL^T$, where $D$ is a diagonal matrix.
   - Use the factorization $A = LDL^T$ to find the Cholesky factorization of $A$, i.e., $A = R^T R$, where $R$ is an upper triangular matrix.

   (c) Explain how you would use the factors $L$ and $U$ to solve the linear equations $A\mathbf{x} = \mathbf{b}$.

13. *SVD* Let $A$ be an $n \times n$ matrix. A right inverse of $A$ is a matrix $B$ such that

$$AB = I,$$

and a left inverse of $A$ is a matrix $C$ such that

$$CA = I.$$

When $A$ is full rank, then it has both right and left inverses and they are equal, i.e., $B = C = A^{-1}$. However, numerically, the left inverse is not necessarily a good right inverse and vice versa, as we will now demonstrate.

Let $A = U\Sigma V^T$, where $U$ and $V$ are $n \times n$ orthogonal matrices

$$U = \begin{pmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \\ | & | & & | \end{pmatrix}, \quad V = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix},$$

and $\Sigma$ is an $n \times n$ diagonal matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

with

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0.$$

(a) Give an explicit expression for $A^{-1}$.

(b) Let $X = A^{-1} + \epsilon \mathbf{v}_n \mathbf{u}_1^T$, where $\epsilon \in \mathbb{R}$. Compute $AX$ and $XA$. Express your answer as a rank-1 perturbation of the identity (i.e., in the form $I + \alpha \mathbf{u}\mathbf{v}^T$ for some scalar $\alpha$, and vectors $\mathbf{u}$, and $\mathbf{v}$).

(c) Given any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, show that $||\mathbf{u}\mathbf{v}^T||_2 = ||\mathbf{u}||_2||\mathbf{v}||_2$. (Hint: recall that the 2-norm of a matrix is given by a its largest singular value).

(d) Use the above result to compute $||AX - I||_2$ and $||XA - I||_2$. What does this say about the accuracy of $X$ as a left and right inverse?