$\boxed{24}$ $A \in \mathbb{R}^{m \times n}$, $m > n$

$$\min_x \| b - Ax \|_2$$

(a) $\| b - Ax \|_2^2$

$$(*) = \| Q^T b - Q^T A x \|_2^2$$

Let $A = QR$

$$\underset{m\times n}{\begin{pmatrix} A \end{pmatrix}} = \underset{m\times m}{\begin{pmatrix} Q \end{pmatrix}} \underset{m\times n}{\overset{R}{\begin{pmatrix} \nabla \\ 0 \end{pmatrix}}}$$

Let $R = \begin{pmatrix} \hat{R} \\ 0 \end{pmatrix}$

Let $C = Q^T b$

$$(*) = \| c - Rx \|_2^2$$

Let $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \begin{matrix} n \\ m-n \end{matrix}$

$$(*) = \left\| \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - \begin{pmatrix} \overset{n}{\hat{R}} \\ 0 \end{pmatrix} x \right\|_2^2$$

$$= \| c_1 - \hat{R} x \|_2^2 + \| c_2 \|_2^2$$

Let $\hat{R} x = c_1$

Solve by backward substitution.

Then x solves to L.S. problem.

(b)

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T} = U\begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} V^T$$

$$\circledast = \|b - Ax\|_2^2 = \|U^T b - U^T A x\|_2^2 \quad \text{(since } U \text{ is orthogonal)}$$

Let $c = U^T b$ (so $c_i = u_i^T b$)

$$\circledast = \|c - \Sigma V^T x\|_2^2$$

$$= \left\| \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \\ c_{r+1} \\ \vdots \\ c_m \end{pmatrix} - \begin{pmatrix} \sigma_1 v_1^T x \\ \sigma_2 v_2^T x \\ \vdots \\ \sigma_r v_r^T x \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\|_2^2$$

$$= \left\| \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix} - \begin{pmatrix} \sigma_1 v_1^T x \\ \vdots \\ \sigma_r v_r^T x \end{pmatrix} \right\|_2^2 + \left\| \begin{pmatrix} c_{r+1} \\ \vdots \\ c_m \end{pmatrix} \right\|_2^2$$

want $\sigma_i v_i^T x = c_i$ for $i = 1, \dots, r$

so choose $\vec{x} = \sum_{i=1}^{r} \frac{c_i}{\sigma_i} \vec{v}_i$

$$\boxed{\vec{x} = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} \vec{v}_i}$$

(c) The solution is ~~that~~ unique in part ~~to~~ (a) since the matrix ~~problem~~ is full rank.

It is not unique in (b) since A is not full-rank.

(d) $b \in \text{Range}(A)$