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(a) A is invertible as it has all singular values > 0.

$$A^{-1} = (U\Sigma V^T)^{-1} = (V^T)^{-1}\Sigma^{-1}U^{-1}$$

$$= V\Sigma^{-1}U^T, \text{ where } \Sigma^{-1} = \begin{pmatrix} \frac{1}{\sigma_1} & & & \\ & \frac{1}{\sigma_2} & & \\ & & \ddots & \\ & & & \frac{1}{\sigma_n} \end{pmatrix}$$

(b)

$$AX = U\Sigma V^T(V\Sigma^{-1}U^T + \varepsilon v_n u_1^T) = I + \varepsilon U\Sigma V^T v_n u_1^T$$

$$= I + \varepsilon U\Sigma \hat{e}_n u_1^T = I + \varepsilon U \sigma_n \hat{e}_n u_1^T, \text{ since } \Sigma\hat{e}_n = \sigma_n e_n$$

$$= \boxed{I + \varepsilon\sigma_n \vec{u}_n \vec{u}_1^T}, \text{ since } U\hat{e}_n = \vec{u}_n$$

$$XA = (A^{-1} + \varepsilon v_n u_1^T)A = I + \varepsilon v_n(u_1^T A)$$

$$= I + \varepsilon \vec{v}_n u_1^T U\Sigma V^T$$

$$= I + \varepsilon \vec{v}_n \hat{e}_1^T \Sigma V^T, \text{ since } u_1^T U = \hat{e}_1^T$$

$$= I + \varepsilon \vec{v}_n \sigma_1 \hat{e}_1^T V^T, \text{ since } \hat{e}_1^T \Sigma = \sigma_1 \hat{e}_1^T$$

$$= \boxed{I + \varepsilon\sigma_1 \vec{v}_n \vec{v}_1^T}, \text{ since } \hat{e}_1^T V^T = \vec{v}_1^T$$

(c) $\|uv^T\|_2 = \left\| \|u\|_2 \frac{u}{\|u\|_2} \|v\|_2 \frac{v^T}{\|v\|_2} \right\|_2$

$$= \|u\|_2 \|v\|_2 \left\| \frac{u}{\|u\|_2} \frac{v^T}{\|v\|_2} \right\|_2$$

$$= \|u\|_2 \|v\|_2 \, \| A \|_2 , \quad = (*)$$

where $A = \frac{u}{\|u\|_2} \frac{v^T}{\|v\|_2}$

The SVD of A is

$$A = \begin{pmatrix} | & | & & | \\ \frac{u}{\|u\|_2} & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \begin{pmatrix} — \frac{v^T}{\|v\|_2} — \\ — v_2^T — \\ \vdots \\ — \vec{v}_n^T — \end{pmatrix}$$

for $\vec{u}_2, \dots, \vec{u}_n$ completing orthonormal basis with $u/\|u\|$

and $\vec{v}_2, \dots, \vec{v}_n$ completing orthonormal basis with $v/\|v\|$

Therefor
$$\sigma_{max}(A) = \sigma_{max}\left(\frac{u}{\|u\|_2} \frac{v^T}{\|v\|_2} \right) = 1$$

So $(*) = \|u\|_2 \|v\|_2$

(d)

$$\|AX - I\|_2 = \|\cancel{I} + \varepsilon \sigma_n \vec{u}_n \vec{u}_1^T - \cancel{I}\|_2$$

$$= |\varepsilon| \sigma_n \|u_n u_1^T\|_2$$

$$= |\varepsilon| \sigma_n$$

$$\|XA - I\|_2 = \|\cancel{I} + \varepsilon \sigma_1 v_n v_1^T - \cancel{I}\|_2$$

$$= |\varepsilon| \sigma_1 \|v_n v_1^T\|_2$$

$$= |\varepsilon| \sigma_1$$

X is more or equally accurate as a right inverse than as a left inverse, because

$$\|AX - I\|_2 = |\varepsilon| \sigma_n \leq |\varepsilon| \sigma_1 = \|XA - I\|_2$$