CS 210
Final

Spring 2016

Name	
Student ID	
Signature	

You may not ask any questions during the exam. If you believe that there is something wrong with a question, write down what you think the question is trying to ask, and answer that.

Question	Points	Score
1	3	
2	3	
3	3	
4	3	
5	3	
6	3	
7	3	
8	3	
9	3	
10	3	
11	4	
12	4	
13	4	
14	4	
15	4	
16	4	
17	4	
18	4	
19	4	
20	4	
21	6	
22	6	
23	9	
24	9	
Total	100	

True/False

For each question, indicate whether the statement is true or false by circling T or F, respectively.

1. (T/F) A system of nonlinear equations has either no solutions, exactly one solution, or infinitely many solutions.
2. (T/F) We can't use Newton's method for nonlinear *systems* of equations because it only makes sense for *scalar* equations.
3. (T/F) The convergence rate of Newton's Method for solving $f(x) = 0$ depends on f.
4. (T/F) If the errors in successive iterations of an algorithm are $10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}...$, then the algorithm is exhibiting quadratic convergence.
5. (T/F) If the errors in successive iterations of an algorithm are $10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}...$, then the algorithm is exhibiting quadratic convergence.
6. (T/F) Given a square matrix A, power iteration, inverse power iteration, and Rayleigh quotient iteration are all algorithms that can be used to find an eigenvalue of A.

For questions 7-10, consider fixed point iteration for finding a point x^* such that $g(x^*) = x^*$.

7. (T/F) When convergent, the convergence rate is always linear.
8. (T/F) The iteration is locally convergent if $|g'(x^*)| < 1$.
9. (T/F) The iteration converges for any starting point if $|g'(x^*)| < 1$.
10. (T/F) Newton's Method for solving $f(x) = 0$ is an example of fixed point iteration, with $g(x) = x - f(x)/f'(x)$.

Multiple Choice

Instructions: For the multiple choice problems, circle exactly one of (a) - (e).

11. Which of the following statements about the Singular Value Decomposition (SVD) are true?

    I. Every real matrix has an SVD.

    II. If a matrix Q is orthogonal, then its singular values are all 1.

    III. The singular value decomposition of a symmetric real matrix is the same as its eigenvalue decomposition.

    (a) I only
    (b) I and II only
    (c) I and III only
    (d) II and III only
    (e) I, II and III

12. Which of the following statements about the Least Squares (LS) problem $\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2$ are true?

    I. If $\mathbf{b} \in \text{Range}(A)$, then the LS problem has a residual of norm 0.

    II. The solution of the LS problems satisfies $A^T A\mathbf{x} = A^T\mathbf{b}$.

    III. The matrix $A^T A$ is always symmetric and positive definite.

    (a) I only
    (b) II only
    (c) I and II only
    (d) II and III only
    (e) I, II and III

13. Which of the following statements about Newton's method for finding a root of a nonlinear equation are true?

    (a) The cost per iteration of the Secant method is greater than that of Newton's method.
    (b) Newton's methods exhibits quadratic convergence for any initial guess $\mathbf{x}_0$.
    (c) Newton's method is an example of a fixed point iteration scheme.
    (d) When Newton's method converges, then it converges with a quadratic convergence rate.
    (e) None of the above.

14. Which of the following statements are true?

    I. Finding the root of a function which is nearly "flat" around the root is a well-conditioned problem.

    II. The bisection method has linear convergence with constant 1/2.

    III. If the errors in successive iterations of an algorithm are $10^{-2}, 10^{-4}, 10^{-6}, ...$, then the algorithm is exhibiting quadratic convergence.

    (a) I only
    (b) II only
    (c) III only
    (d) II and III only
    (e) None

15. Let $A = U\Sigma V^T$ be the Singular Value Decomposition (SVD) of the matrix A and let A^+ denote the pseudoinverse of A. Which of the following statements are true?

    I. The SVD reveals the rank of a matrix.

    II. $A^+ = U\Sigma^+ V^T$ where Σ^+ is the pseudoinverse of Σ.

    III. The rank of A is the same as the rank of A^+.

    (a) I only
    (b) III only
    (c) I and II only
    (d) I and III only
    (e) I, II and III

16. Which of the following statements about the Least Squares (LS) problem $\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2$ are true?

    I. The solution of the LS problem always exists.

    II. The solution of the LS problem is always unique.

    III. If A is invertible, then the solution to the LS problem is $\mathbf{x} = A^{-1}\mathbf{b}$.

    (a) I only
    (b) III only
    (c) I and II only
    (d) I and III only
    (e) I, II and III

17. Which of the following statements are true?

    I. An eigenvector corresponding to a given eigenvalue is unique.

    II. Scaling a matrix by a constant c will scale its eigenvalues by that constant.

    III. If a matrix has an eigenvalue of 0, then it is not invertible.

    (a) I only
    (b) II only
    (c) III only
    (d) II and III only
    (e) I, II and III

18. Which of the following statements are true?

    I. λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

    II. If λ is an eigenvalue of A, then $|\lambda| <= ||A||_2$.

    III. If A is a symmetric positive definite matrix, then all its eigenvalues are distinct.

    (a) I only
    (b) II only
    (c) III only
    (d) I and II only
    (e) I and III only

19. Consider solving $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$. Let $J_g(\mathbf{x}) = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{x})$ be the the Jacobian matrix of $\mathbf{g}$. Which of the following statements are true?

    I. If $\mathbf{g}$ is a linear function, then $J_g(\mathbf{x})$ is a constant matrix.

    II. If $\mathbf{g}$ is a linear function, then $J_g(\mathbf{x}) = 0$.

    III. Applying Newton's method, $\mathbf{s}_k = -J_g^{-1}(\mathbf{x}_k)\mathbf{g}(\mathbf{x}_k)$ is the Newton step such that $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{s}_k$.

    (a) I only
    (b) II only
    (c) III only
    (d) I and III only
    (e) II and III only

20. Consider an unconstrained minimization problem where we are seeking a minimizer $\mathbf{x}^*$ of a function $f(\mathbf{x})$. Which of the following statements are true?

    I. The negative gradient of f, $-\nabla f(\mathbf{x})$, points in a "downhill" direction of f.

    II. A critical point $\mathbf{x}^*$ of f is a minimizer of f if the Hessian matrix $H_f(\mathbf{x}^*)$ is negative definite.

    III. A necessary condition for f to have a minimum at $\mathbf{x}^*$ is that $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

    (a) I only
    (b) II only
    (c) III only
    (d) I and III only
    (e) II and III only

Written Response

21. *Nonlinear Equations: Newton's Method.* Consider the system of equations

$$\begin{aligned} x^2 - y^2 &= 0 \\ 2xy &= 1 \end{aligned}$$

Carry out one iteration of Newton's Method for finding a solution to this system, with starting value $\mathbf{x}_0 = (0, 1)^T$.

22. *Optimization.* Consider the function

$$\phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x} + c,$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric.

(a) What are the critical points of ϕ?

(b) How would you classify the critical points of ϕ as maxima, minima or saddle points?

23. *Singular Value Decomposition.* Let A be an $n \times n$ matrix. A right inverse of A is a matrix B such that

$$AB = I,$$

and a left inverse of A is a matrix C such that

$$CA = I.$$

When A is full rank, then it has both right and left inverses and they are equal, i.e., $B = C = A^{-1}$. However, numerically, the left inverse is not necessarily a good right inverse and vice versa, as we will now demonstrate.

Let $A = U\Sigma V^T$, where U and V are $n \times n$ orthogonal matrices

$$U = \begin{pmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \\ | & | & & | \end{pmatrix}, \quad V = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & & | \end{pmatrix},$$

and Σ is an $n \times n$ diagonal matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

with

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0.$$

(a) Show that A is invertible and give an explicit expression for A^{-1}.

(b) Let $X = A^{-1} + \epsilon \mathbf{v}_n \mathbf{u}_1^T$, where $\epsilon \in \mathbb{R}$. Compute AX and XA. Express your answer as a rank-1 perturbation of the identity (i.e., in the form $I + \alpha \mathbf{u}\mathbf{v}^T$ for some scalar α, and vectors $\mathbf{u}$, and $\mathbf{v}$).

(c) Given any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, show that $||\mathbf{u}\mathbf{v}^T||_2 = ||\mathbf{u}||_2 ||\mathbf{v}||_2$. (Hint: recall that the 2-norm of a matrix is given by a its largest singular value).

(d) Use the above result to compute $||AX - I||_2$ and $||XA - I||_2$. What does this say about the accuracy of X as a left and right inverse?

24. *Least Squares.* Let $A \in \mathbb{R}^{m \times n}$, where $m > n$. Consider the least squares (LS) problem

$$\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2.$$

(a) Assume A has full rank. Show how you would use the QR decomposition $A = Q \begin{pmatrix} R \\ 0 \end{pmatrix}$ to solve the LS problem.

(b) Now assume A is rank-deficient with rank $r < n$. Show how you would use the Singular Value Decomposition $A = U\Sigma V^T$, with $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$, to solve the LS problem.

(c) In parts (a) and (b) is the solution unique? Why or why not?

(d) What does it say about $\mathbf{b}$ if $\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2 = 0$?