✓

2.4 Solving Linear Systems $Ax = b$

(1) transform into a linear system that is easy to solve.

e.g., triangular system

(3)

$$\begin{pmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{pmatrix} \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} = \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix}$$

(Q) How would you solve it?

If M is non-singular, (A and) then solution is not affected by

$$MA\cancel{z} = Mb$$

(2) because

$$Ax = b$$

$$\cancel{z} = (MA)^{-1} Mb = A^{-1} M^{-1} M b = A^{-1} b \checkmark$$

$$= x \checkmark$$

Ex. 2.9 Permutation Matrix $P^{-1} = P^{T}$

- identity with rows + cols permuted

row permutation

cols permutation

(row permutation)

$$PAx = Pb$$ reorders the equation

x unchanged

(col permutations)

$$AP\cancel{z} = b \Rightarrow \cancel{z} = (AP)^{-1} \cancel{b} = P^{T} A^{-1} b$$

$$AP(P^{T}x) = b \qquad = P^{T} x \text{ row perm}$$

Ex. 2.10 Diagonal Scaling

$DAx = Db$ $\quad DAz = Db$ row scaling — doesn't change exact soln, but

$ADz = b$ column scaling

$$\Rightarrow z = (AD)^{-1} b = D^{-1} A^{-1} b = D^{-1} x$$

$$ADD^{-1} z = b$$

$\boxed{\text{can change cond. \#}}$

$\boxed{\text{does change numerical accuracy}}$

Lower Triangular Matrix

$$L = \left(\text{◺}\ 0\right) = \begin{pmatrix} x & & & \\ x & x & & \\ x & x & x & \\ x & x & x & x \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \vec{l}_1 & \vec{l}_2 & \cdots & \vec{l}_n \\ | & | & & | \end{pmatrix}$$

$\vec{l}_j$ = j^{th} column of L

<u>Solve</u>

$$L\vec{x} = \vec{b}$$

$$\vec{l}_1 x_1 + \vec{l}_2 x_2 + \ldots + \vec{l}_n x_n = \vec{b}$$ "column view"

$$\begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ l_{31} & l_{32} & l_{33} & \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$ 4x4 system

$$\boxed{x_1 = b_1 / l_{11}}$$

now x_1 is known, so subtract it off:

$$\begin{pmatrix} l_{22} & & \\ l_{32} & l_{33} & \\ l_{42} & l_{43} & l_{44} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_2 - x_1 l_{21} \\ b_3 - x_1 l_{31} \\ b_4 - x_1 l_{41} \end{pmatrix}$$ 3x3 system

$$x_2 = (b_2 - x_1 l_{21}) / l_{22}$$

now x_2 is known, so subtract it off.

$$\begin{pmatrix} l_{33} & \\ l_{43} & l_{44} \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_3 - x_1 l_{31} - x_2 l_{32} \\ b_4 - x_1 l_{41} - x_2 l_{42} \end{pmatrix}$$ 2x2 system ...

Upper Triangular Matrix U

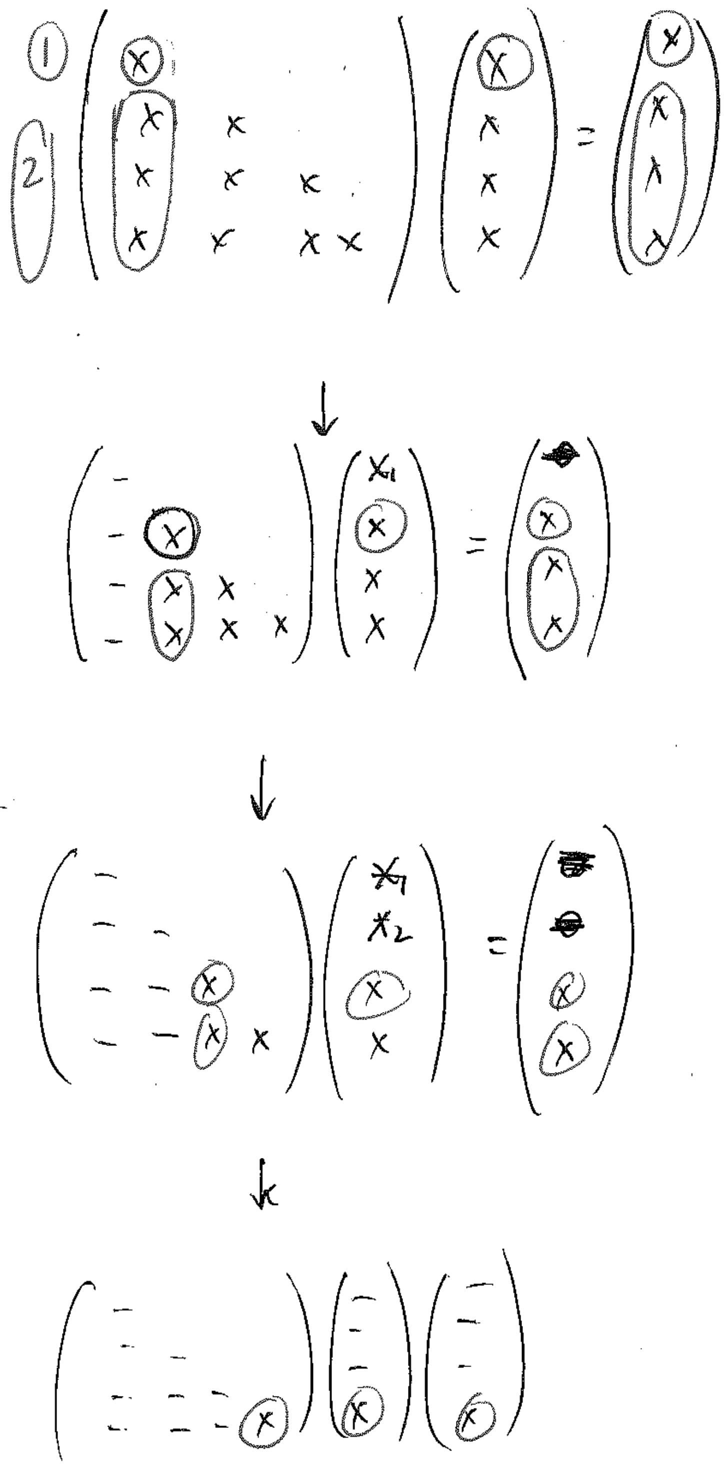
$$U = \begin{pmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{pmatrix} = \begin{pmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \\ | & | & & | \end{pmatrix}$$

$$U\vec{x} = \vec{b}$$

$$\vec{u}_1 x_1 + \vec{u}_2 x_2 + \dots + \vec{u}_n x_n = \vec{b}$$

Forward Substitution

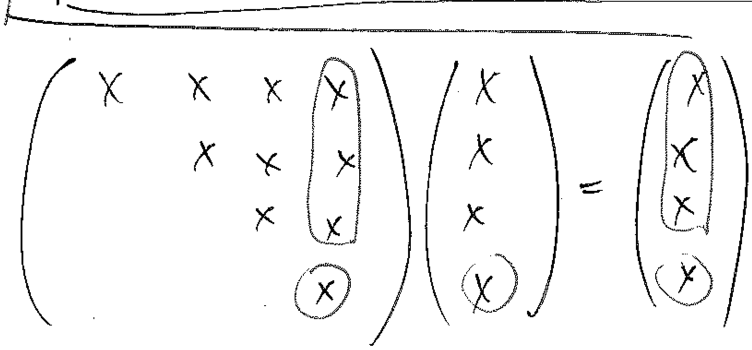
```
for j = 1 ... n
    if l_jj == 0 stop
(1) x_j = b_j / l_jj
(2) for i = j+1 ... n
        b_i = b_i - l_ij x_j
    end
end
```



Backward Substitution

```
for j = n ... 1
    if u_jj == 0 then stop
    x_j = b_j / u_jj

    for i = j-1 ... 1
        b_i = b_i - u_ij x_j
    end
end
```



$$\begin{pmatrix} 1 & 0 \\ -\frac{a_2}{a_1} & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ -\frac{a_2}{a_1}\cdot a_1 + a_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \end{pmatrix}$$

eliminate element 2 using multiple of element 1

Elementary Elimination Matrix or Gauss Transform

$$M_k =$$

$$\begin{array}{c|c} & k \\ k & \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & -m_{k+1} & 1 & & \\ & & \vdots & & \ddots & \\ & & -m_n & & & 1 \end{array}\right) \end{array} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$m_i = \frac{a_i}{a_k}, \quad i = k+1 \ldots n$$

leave first k elements unchanged

a_k: pivot

$j > k$

$$\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & -m_{j+1} & 1 & & \\ & & \vdots & & \ddots & \\ & & -m_n & & & 1 \end{array}\right)\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & -m_{k+1} & 1 & & \\ & & \vdots & & \ddots & \\ & & -m_n & & & 1 \end{array}\right)$$

$$\vec{e}_j^{\,T} \vec{m}_k = (0 \cdots 0 1 0 \cdots 0)\begin{pmatrix} 0 \\ \vdots \\ m_{k+1} \\ \vdots \\ m_n \end{pmatrix} = m_j$$

X $M_j M_k = (I - \vec{m}_j \vec{e}_j^{\,T})(I - \vec{m}_k \vec{e}_k^{\,T}) = I - m_j e_j^T - m_k e_k^T + \vec{m}_j (e_j^T m_k) e_k^T$

$= I - \vec{m}_j \vec{e}_j^{\,T} - \vec{m}_k \vec{e}_k^{\,T} + m_j \vec{m}_j \vec{e}_k^{\,T}$

✓ $M_k M_j = (I - \vec{m}_k \vec{e}_k^{\,T})(I - \vec{m}_j \vec{e}_j^{\,T}) = I - \vec{m}_k \vec{e}_k^{\,T} - \vec{m}_j \vec{e}_j^{\,T} + \vec{m}_k (\underbrace{\vec{e}_k^{\,T} \vec{m}_j}_{0}) \vec{e}_j^{\,T}$

$= I - \vec{m}_k \vec{e}_k^{\,T} - \vec{m}_j \vec{e}_j^{\,T}$

✓ $M_k^{-1} M_j^{-1} = (I + m_k e_k^T)(I + m_j e_j^T) = I + m_k e_k^T + m_j e_j^T$

Lecture 20. Gaussian Elimination. $\boxed{\text{LECTURE 3}}$ ✓

$$\underbrace{M_{n-1} \cdots M_2 M_1}_{L^{-1}} A = U$$

$$A = LU \qquad L \text{ "unit lower triangular"}$$

General Formula for M_k

$$M_k a = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & -m_{k+1} & 1 & \\ & & \vdots & & \ddots \\ & & -m_n & & & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

k^{th} column

$$m_i = \frac{a_i}{a_k}$$

a_k is the "pivot"

- M_k is unit lower triangular
- M_k is of the form $I - \vec{m}_k \vec{e}_k^{\,T}$, $\vec{m}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{pmatrix}$

(*) • M_k^{-1} is really easy to compute:

$$M_k^{-1} = I + \vec{m}_k \vec{e}_k^{\,T} \qquad \text{(check)}$$

(*) • L is really easy to compute

$$L^{-1} = M_{n-1} M_{n-2} \cdots M_2 M_1$$

$$L = (\quad " \quad)^{-1}$$

$$= M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1}$$

$$M_j^{-1} M_k^{-1} = (I + m_j e_j^T)(I + m_k e_k^T) = I + m_j e_j^T + m_k e_k^T + m_j e_j^T m_k e_k^T$$

but $e_j^T m_k = 0$ when $j < k$.

And all terms in product $m_j e_j^T m_k e_k^T$ will have $j<k$.
Therefore

$$L = M_1^{-1} M_2^{-1} \cdots M_{n-1}^{-1}$$

$$\boxed{L = I + m_1 e_1^T + m_2 e_2^T + \dots + m_{n-1} e_{n-1}^T}$$

i.e., $$L = \begin{bmatrix} 1 & & & & \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ m_{n1} & m_{2n} & \cdots & m_{n-1n} & 1 \end{bmatrix}$$

Algorithm <u>Gaussian Elimination</u>
<u>LU Factorization</u>

```
for k = 1, ..., n-1
    if a_kk = 0  stop
    for i = k+1 ... n
        m_ik = a_ik / a_kk
    end
    for j = k+1 ... n
        for i = k+1 ... n
(*)         a_ij = a_ij - m_ik a_kj
        end
    end
```

[for each column (except last)]

[compute elements of L, L_k]

[update to find elements of U]
(upper $k \times n$ elements untouched because M_k^{-1} is the identity there)

In place: write m_{ik} directly into lower part of matrix.
replace m_{ik} with a_{ik} above

(*) $$AB = \sum_{i=1}^{n} \vec{a}_i \vec{b}_i^T$$

$[|||][\equiv]$

4×4 LU Factorization

$$\left(\begin{array}{c|ccc} X & X & X & X \\ \hline X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{array}\right)$$

$$\underbrace{\begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & & 1 & \\ m_{41} & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -m_{21} & 1 & & \\ -m_{31} & & 1 & \\ -m_{41} & & & 1 \end{pmatrix} \left(\begin{array}{c|ccc} X & X & X & X \\ \hline X & X & X & X \\ X & X & X & X \\ X & X & X & X \end{array}\right)}$$

$$\begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & & 1 & \\ m_{41} & & & 1 \end{pmatrix} \left(\begin{array}{c|ccc} X & X & X & X \\ \hline 0 & X & X & X \\ 0 & X & X & X \\ 0 & X & X & X \end{array}\right) = M_1^{-1} M_1 A$$

$$M_1^{-1} M_2^{-1} M_2 M_1 A$$

$$\underbrace{\begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & & 1 & \\ m_{41} & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & m_{32} & 1 & \\ & m_{42} & 0 & 1 \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & -m_{32} & 1 & \\ & -m_{42} & & 1 \end{pmatrix} \left(\begin{array}{cc|cc} X & X & X & X \\ 0 & X & X & X \\ \hline 0 & X & X & X \\ 0 & X & X & X \end{array}\right)}$$

$$\begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & m_{32} & 1 & \\ m_{41} & m_{42} & 0 & 1 \end{pmatrix} \left(\begin{array}{cc|cc} X & X & X & X \\ 0 & X & X & X \\ \hline 0 & 0 & X & X \\ 0 & 0 & X & X \end{array}\right)$$

$$\begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & m_{32} & 1 & \\ m_{41} & m_{42} & m_{43} & 1 \end{pmatrix} \begin{pmatrix} X & X & X & X \\ 0 & X & X & X \\ 0 & 0 & X & X \\ 0 & 0 & 0 & X \end{pmatrix} \qquad \left(M_1^{-1} M_2^{-1} M_3^{-1}\right)\left(M_3 M_2 M_1 A\right) = L\,U$$