Homework 2
CS 210

1. (Trefethen&Bau 2.6) If $\mathbf{u}$ and $\mathbf{v}$ are $m-$vectors, the matrix $A = I + \mathbf{u}\mathbf{v}^T$ is known as a rank-one pertubation of the identity. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha\mathbf{u}\mathbf{v}^T$ for some scalar α, and give an expression for α. For what $\mathbf{u}$ and $\mathbf{v}$ is A singular? If it is singular, what is null(A)?

2. (T&B 4.1) Determine SVDs of the following matrices (by hand calculation):

   (a) $\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

3. Let A be an $m \times n$ singular matrix of rank r with SVD

$$A = U\Sigma V^T = \left(\begin{array}{c|c|c|c} & & & \\ \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_m \\ & & & \end{array}\right) \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \vdots \\ \hline \mathbf{v}_n^T \end{pmatrix}$$

$$= \begin{pmatrix} \hat{U} & \tilde{U} \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \begin{pmatrix} \hat{V}^T \\ \tilde{V}^T \end{pmatrix}$$

   where $\sigma_1 \geq \ldots \geq \sigma_r > 0$, $\hat{U}$ consists of the first r columns of U, $\tilde{U}$ consists of the remaining $m - r$ columns of U, $\hat{V}$ consists of the first r columns of V, and $\tilde{V}$ consists of the remaining $n - r$ columns of V. Give bases for the spaces range(A), null(A), range(A^T) and null(A^T) in terms of the components of the SVD of A, and a brief justification.

4. Show that for an $m \times n$ matrix of full column rank n, the matrix $A(A^TA)^{-1}A^T$ is an orthogonal projector onto range(A). Hint: use the SVD of A.

5. Consider the least squares problem $\min_{\mathbf{x}} ||\mathbf{b} - A\mathbf{x}||_2^2$. Using the SVD of A given in Problem 3, show that

   (a) the least squares solution $\mathbf{x}$ satisfies

$$\mathbf{x} = \alpha_1\mathbf{v}_1 + \ldots + \alpha_r\mathbf{v}_r, \quad \alpha_i = \frac{\mathbf{u}_i^T\mathbf{b}}{\sigma_i}$$

   (b) the minimum residual $\mathbf{r} = \mathbf{b} - A\mathbf{x}$ satisfies $||\mathbf{r}||_2^2 = \sum_{i=r+1}^{m}(\mathbf{u}_i^T\mathbf{b})^2$.