Homework 1
CS 210

1. (Heath 1.2) What are the approximate absolute and relative errors in approximating π by each of the following quantities?

   (a) 3
   (b) 3.14
   (c) 22/7

2. (Heath 1.3) If a is an approximate value for a quantity whose true value is t, and a has a relative error r, prove from the definitions of these terms that $a = t(1+r)$.

3. (Heath 1.7) A floating point number system is characterized by four integers: the base β, the precision p, and the lower and upper limits L and U of the exponent range.

   (a) If $\beta = 10$, what are the smallest values of p and U, and the largest value of L, such that both 2365.27 and 0.0000512 can be represented *exactly* in a *normalized* floating-point system?
   (b) How would your anser change if the system is not normalized, i.e., if gradual underflow is allowed?

4. (Trefethen & Bau 13.1) Between an adjacent pair of nonzero IEEE single precision real numbers, how many IEEE doule precision numbers are there?

5. (Heath 2.4a) Show that the following matrix is singular.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$$

6. (Heath 2.8) Let A and B be any two $n \times n$ matrices.

   (a) Prove that $(AB)^T = B^T A^T$.
   (b) If A and B are both non-singular, prove that $(AB)^{-1} = B^{-1}A^{-1}$.

7. (T&B 1.1) Let B be 4×4 matrix to which we apply the following operations:
   (1) double column 1,
   (2) halve row 3,
   (3) add row 3 to row 1,
   (4) interchange columns 1 and 4,
   (5) subtract row 2 from each of the other rows,
   (6) replace column 4 by column 3,
   (7) delete column 1 (so that the column dimension is reduced by 1)

   (a) Write the result as a produce of eight matrices.
   (b) Write it again as a product ABC (same B) of three matrices.

8. Let $\mathbf{x} \in \mathbb{R}^n$. Two vector norms, $||\mathbf{x}||_a$ and $||\mathbf{x}||_b$, are *equivalent* if $\exists\, c, d \in \mathbb{R}$ such that

$$c||\mathbf{x}||_b \le ||\mathbf{x}||_a \le d||\mathbf{x}||_b.$$

Matrix norm equivalence is defined analogously to vector norm equivalence, i.e., $||\cdot||_a$ and $||\cdot||_b$ are equivalent if $\exists\, c, d$ s.t. $c||A||_b \le ||A||_a \le d||A||_b$.

(a) Let $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$. For each of the following, verify the inequality and give an example of a non-zero vector or matrix for which the bound is achieved (showing that the bound is tight):

i. $||\mathbf{x}||_\infty \le ||\mathbf{x}||_2$

ii. $||\mathbf{x}||_2 \le \sqrt{n}||\mathbf{x}||_\infty$

iii. $||A||_\infty \le \sqrt{n}||A||_2$

iv. $||A||_2 \le \sqrt{n}||A||_\infty$

This shows that $||\cdot||_\infty$ and $||\cdot||_2$ are equivalent, and that their induced matrix norms are equivalent.

(b) Prove that the equivalence of two vector norms implies the equivalence of their induced matrix norms.