# CS130 : Computer Graphics
## Curves

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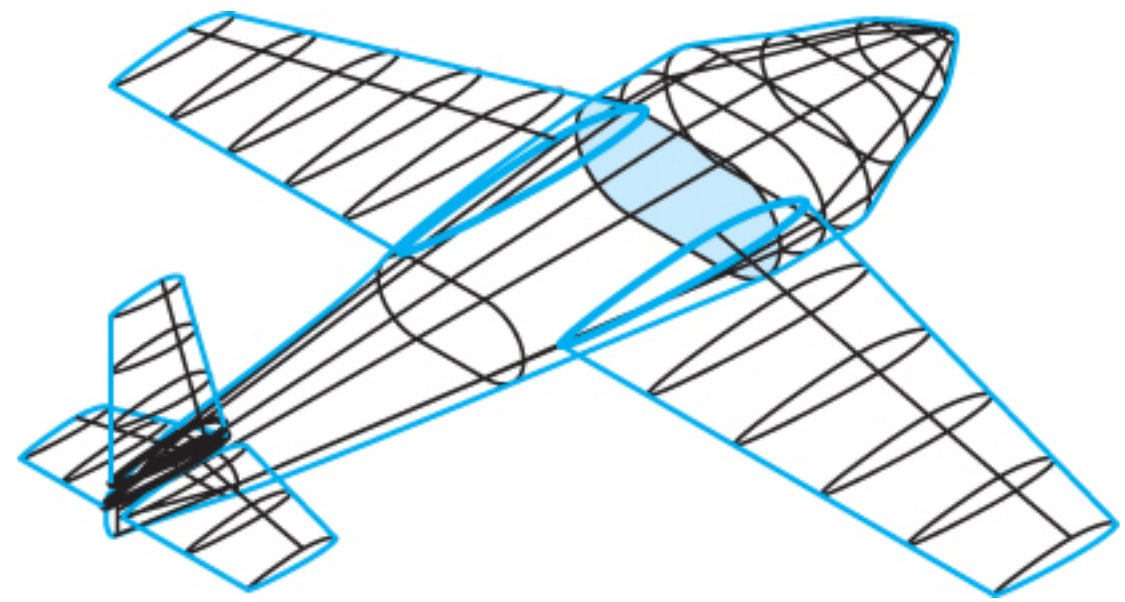
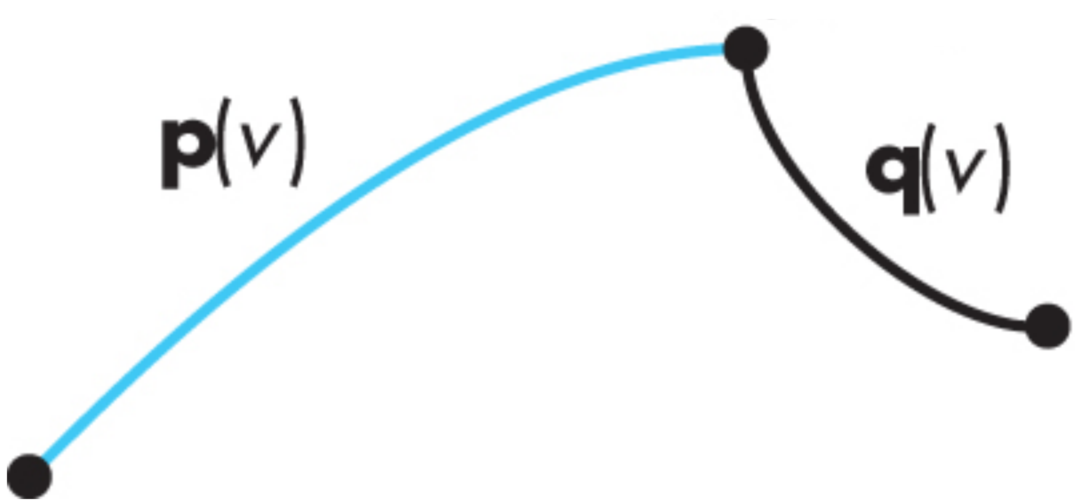
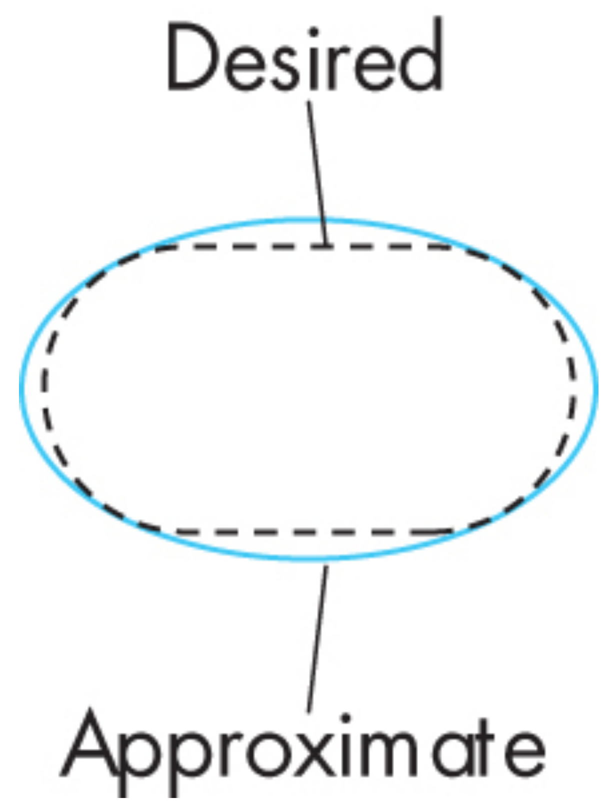
# Design considerations

- local control of shape
  - design each segment independently
- smoothness and continuity
- ability to evaluate derivatives
- stability
  - small change in input leads to small change in output
- ease of rendering





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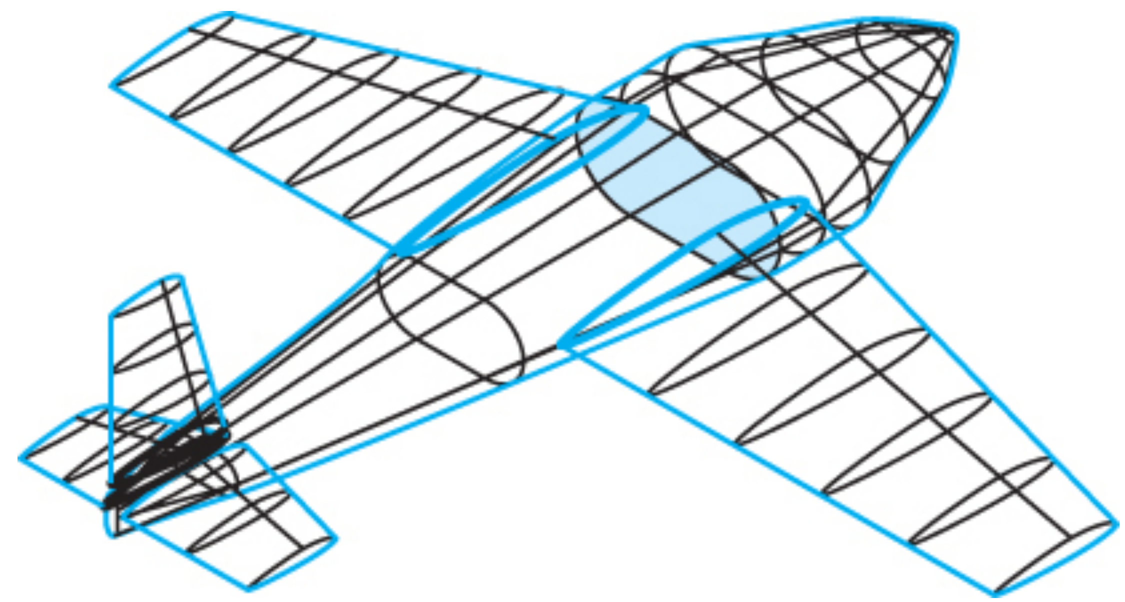

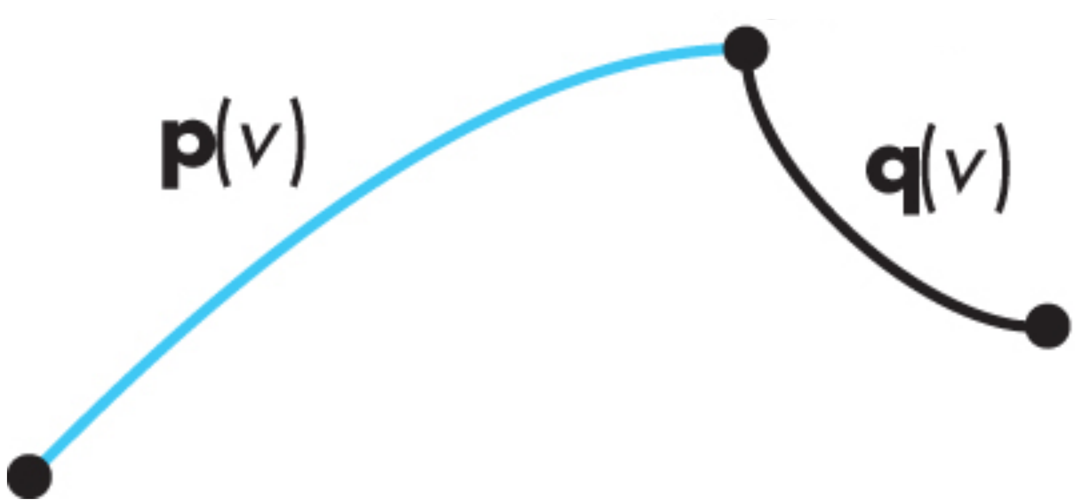


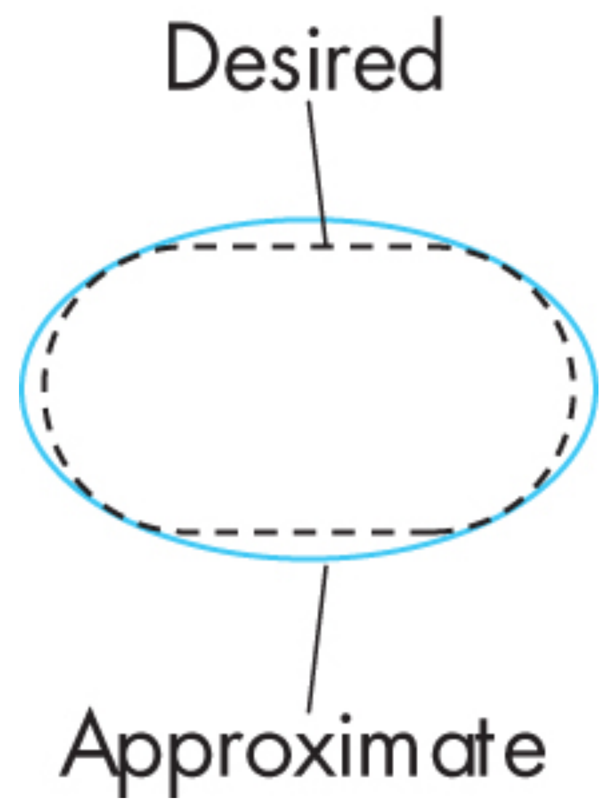


approximate out of a number of wood strips

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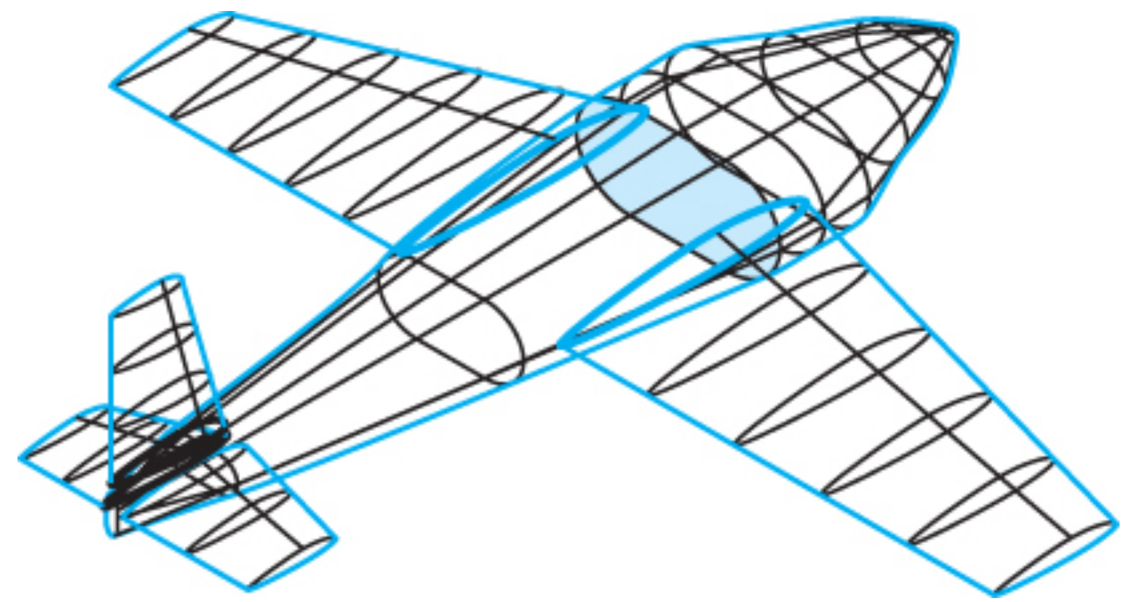
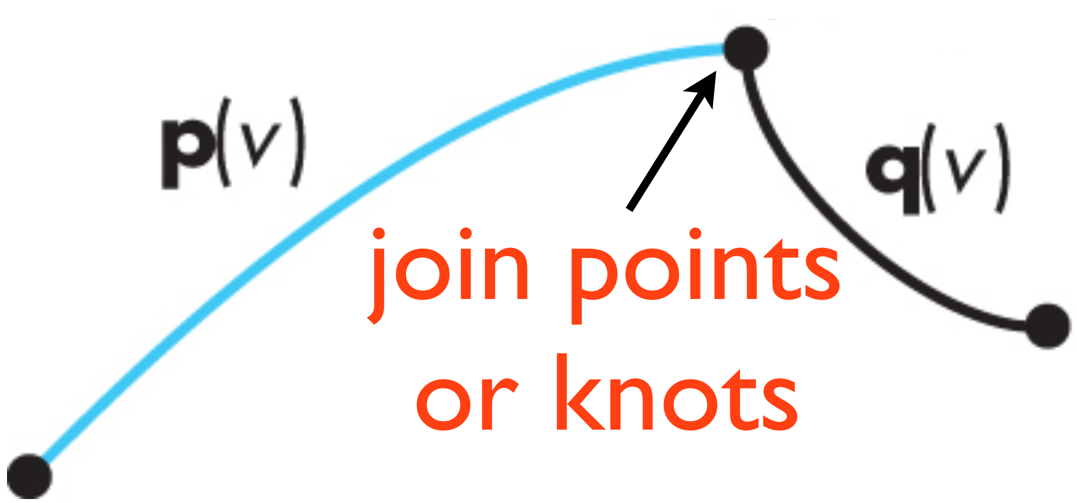
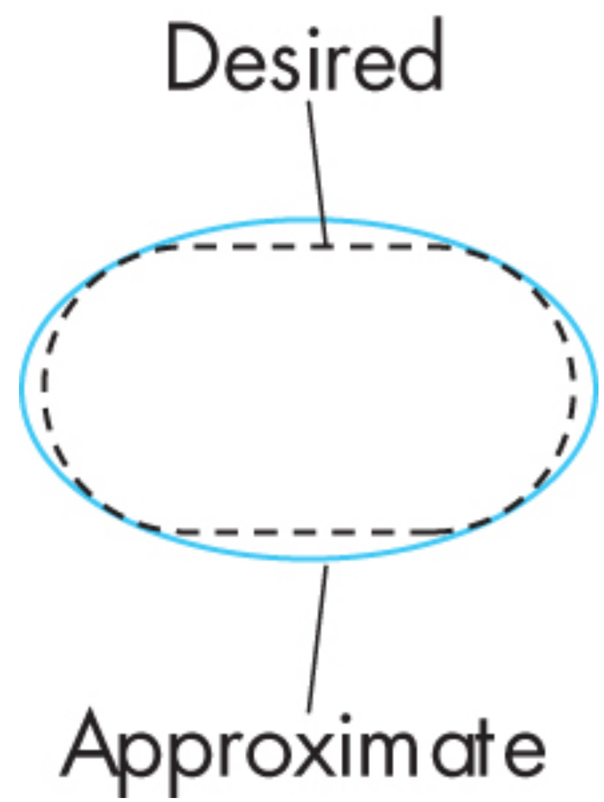




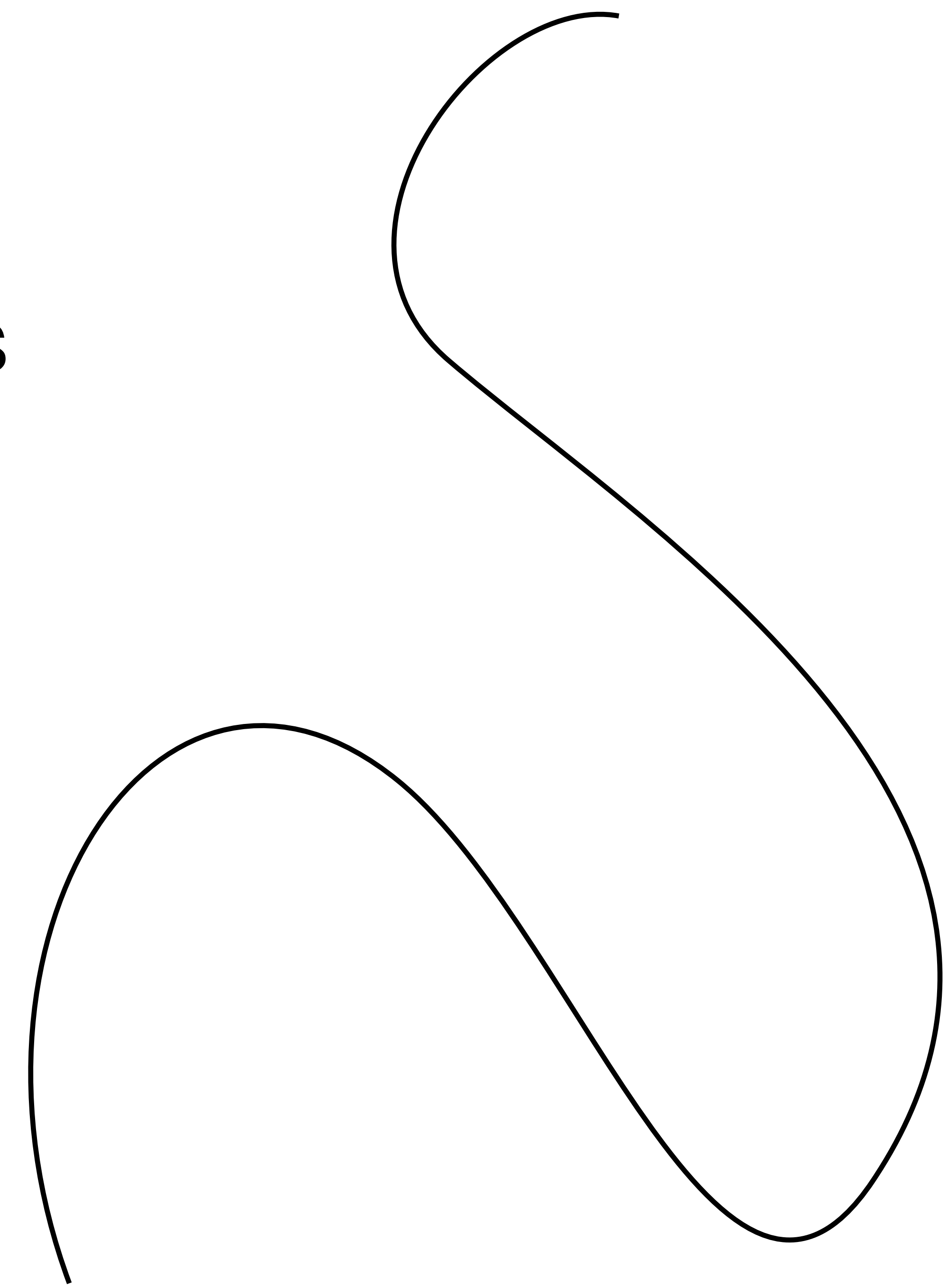




approximate out of a number of wood strips

# What is a curve?

intuitive idea:
  draw with a pen
  set of points the pen traces

may be 2D, like on paper
or 3D, *space curve*



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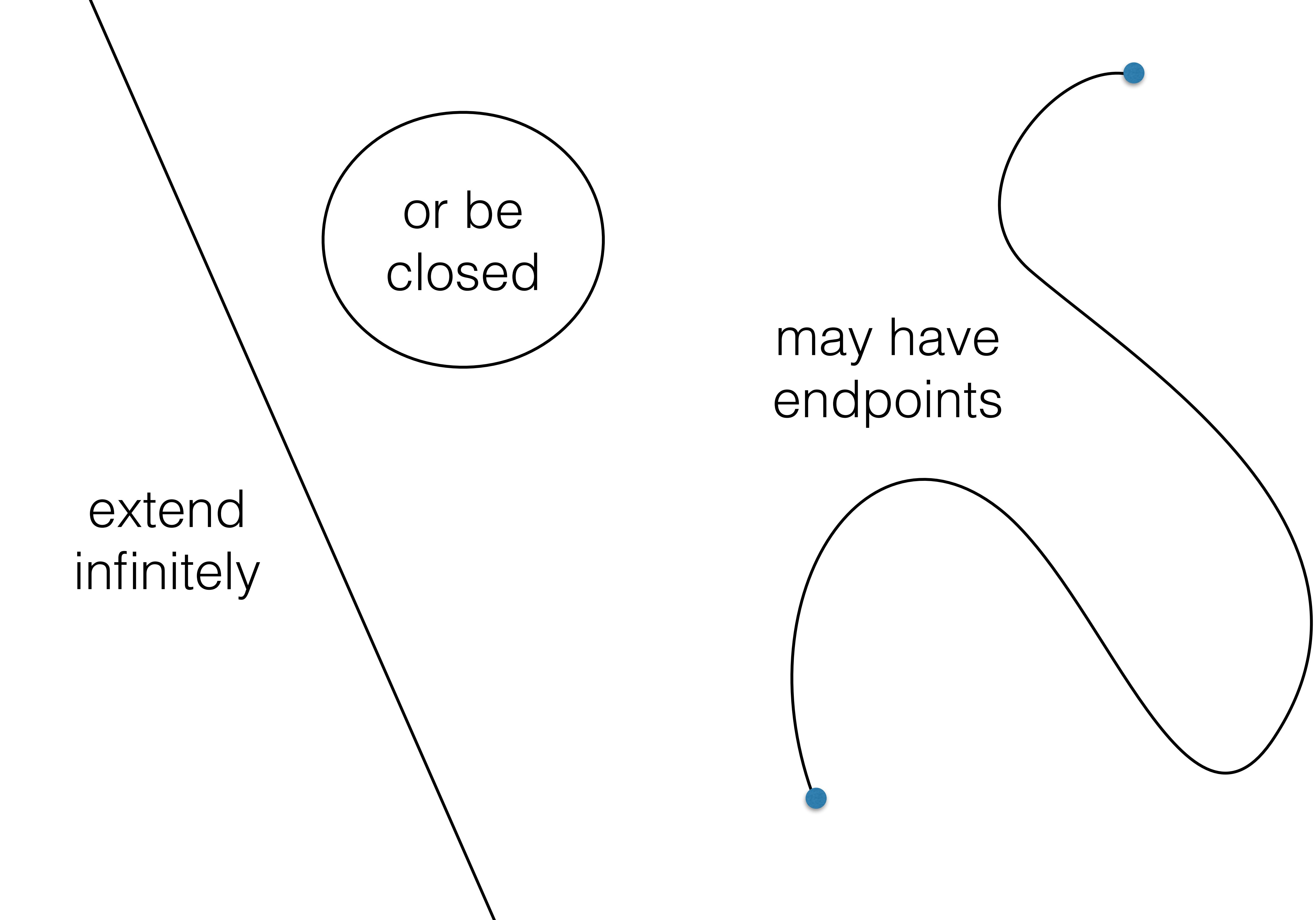
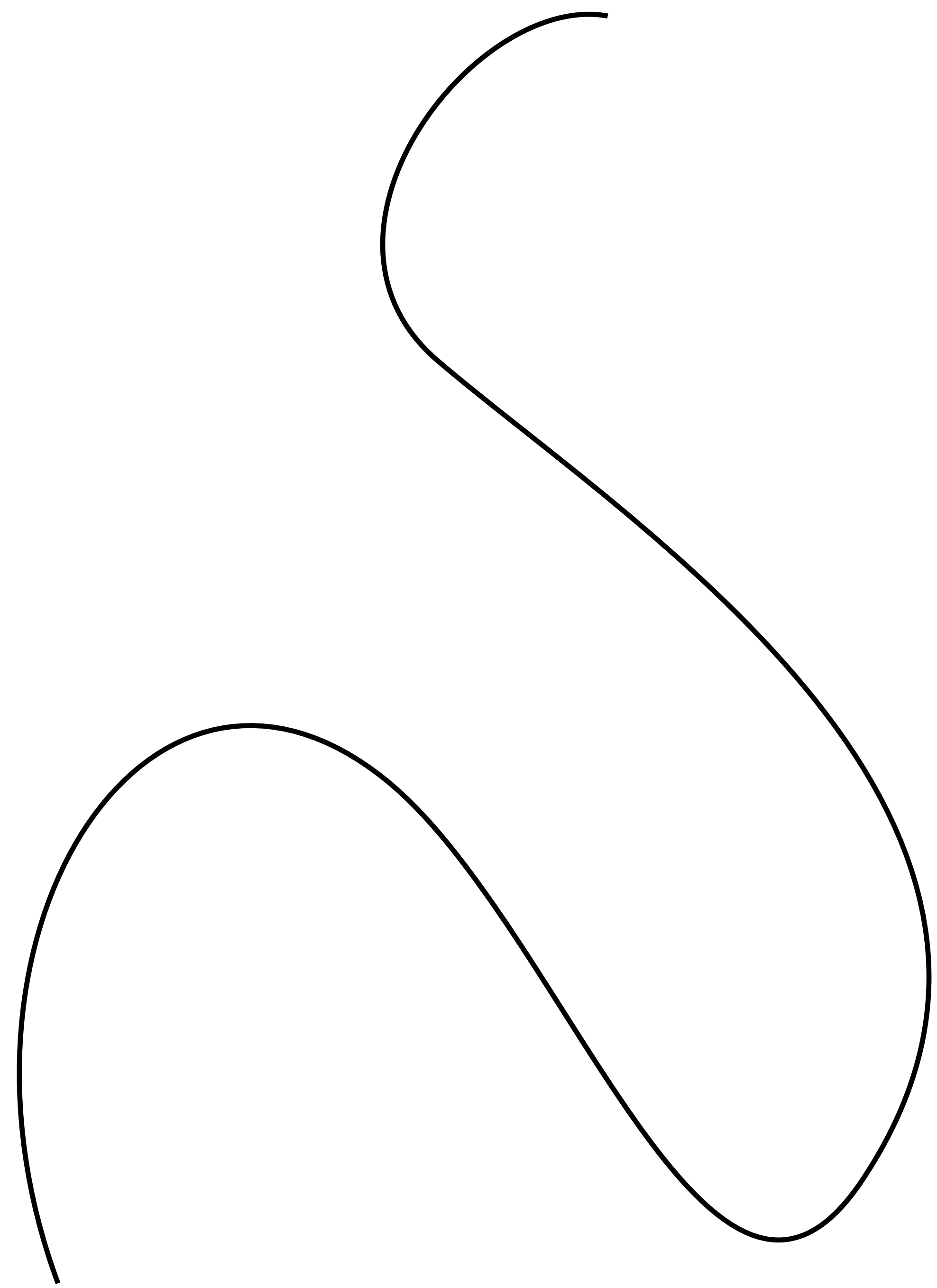


# How do we specify a curve?

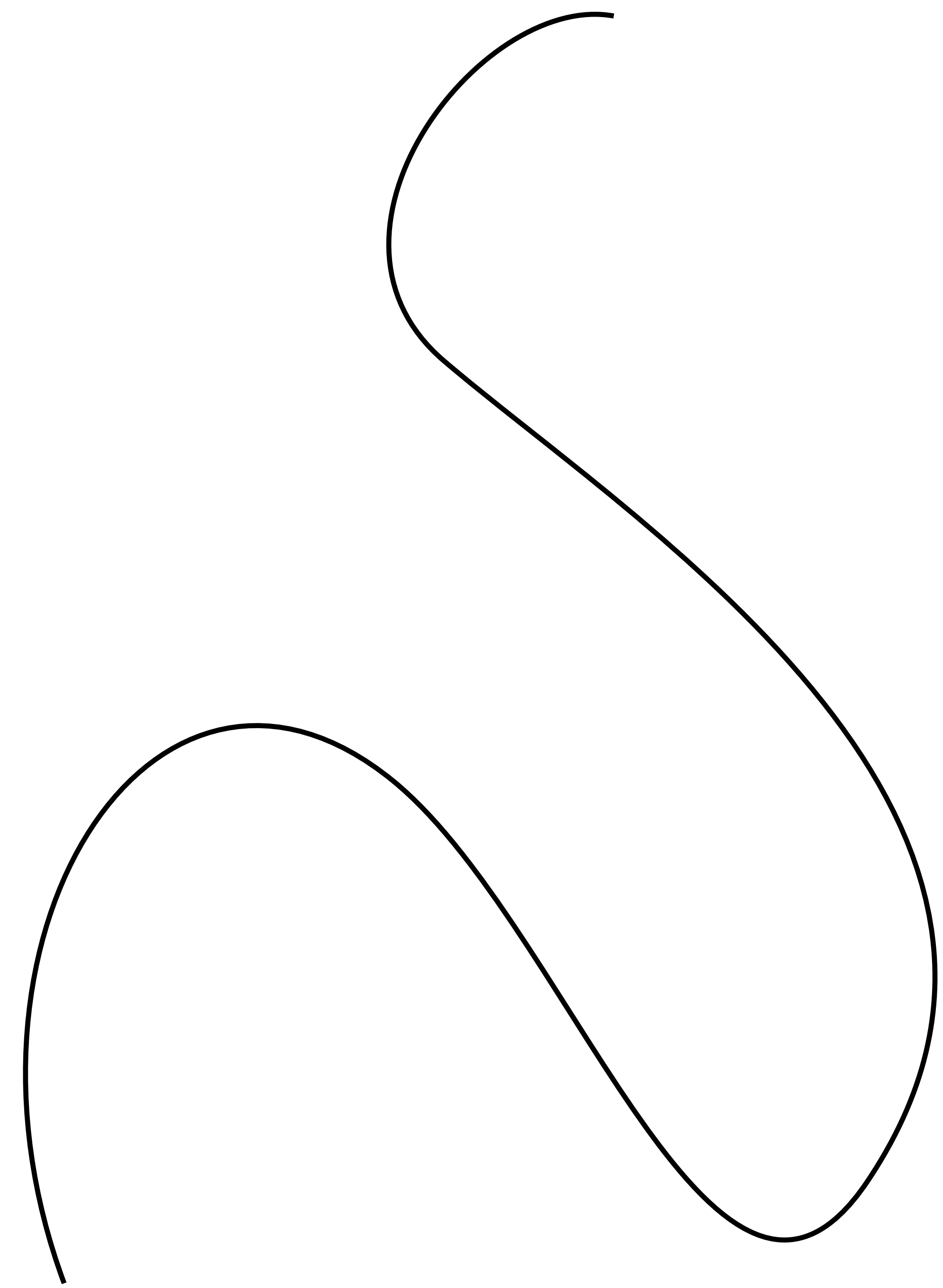


# How do we specify a curve?

*Implicit*

(2D) $f(x,y) = 0$

test if $(x,y)$ is on the curve



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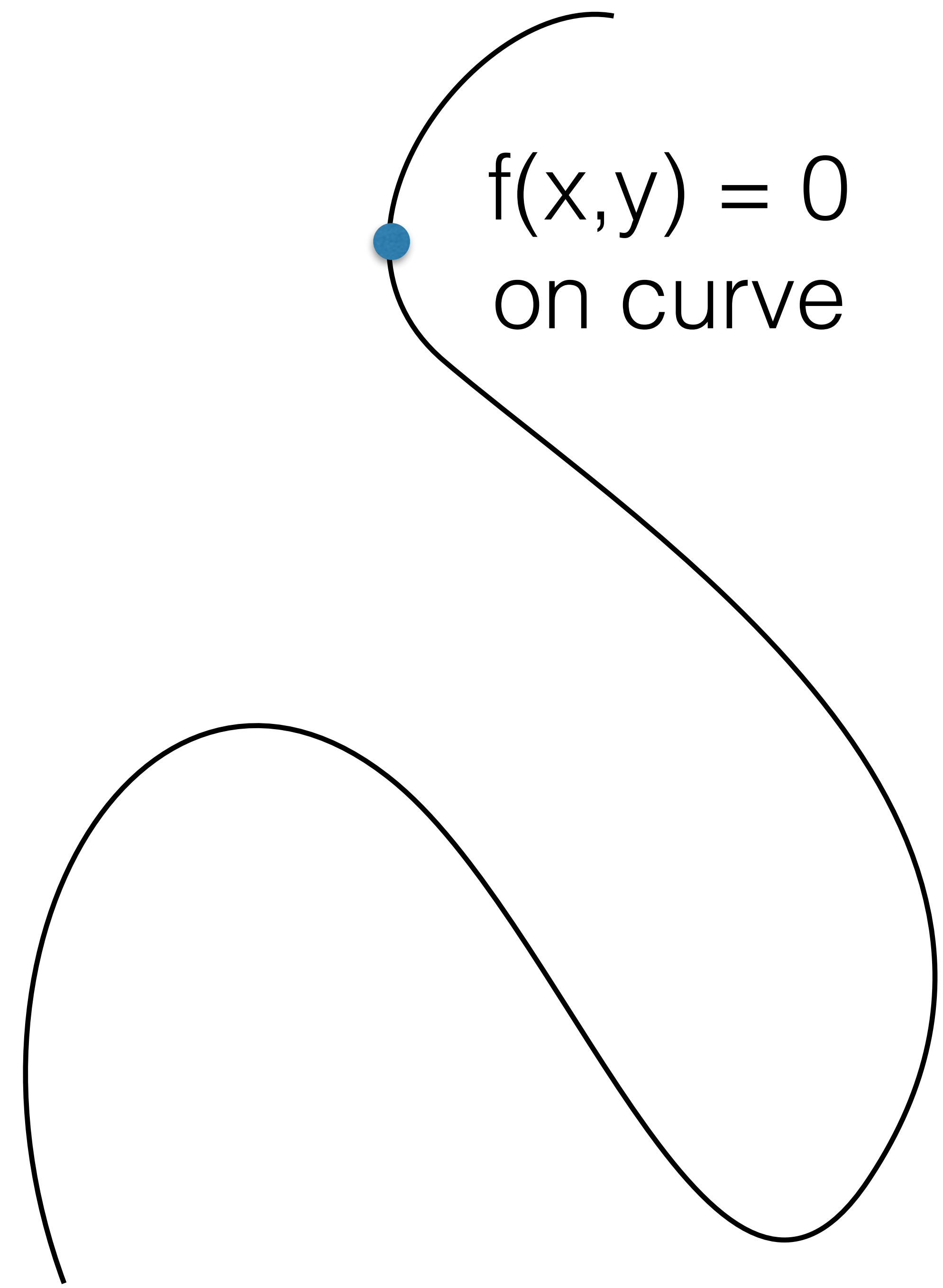

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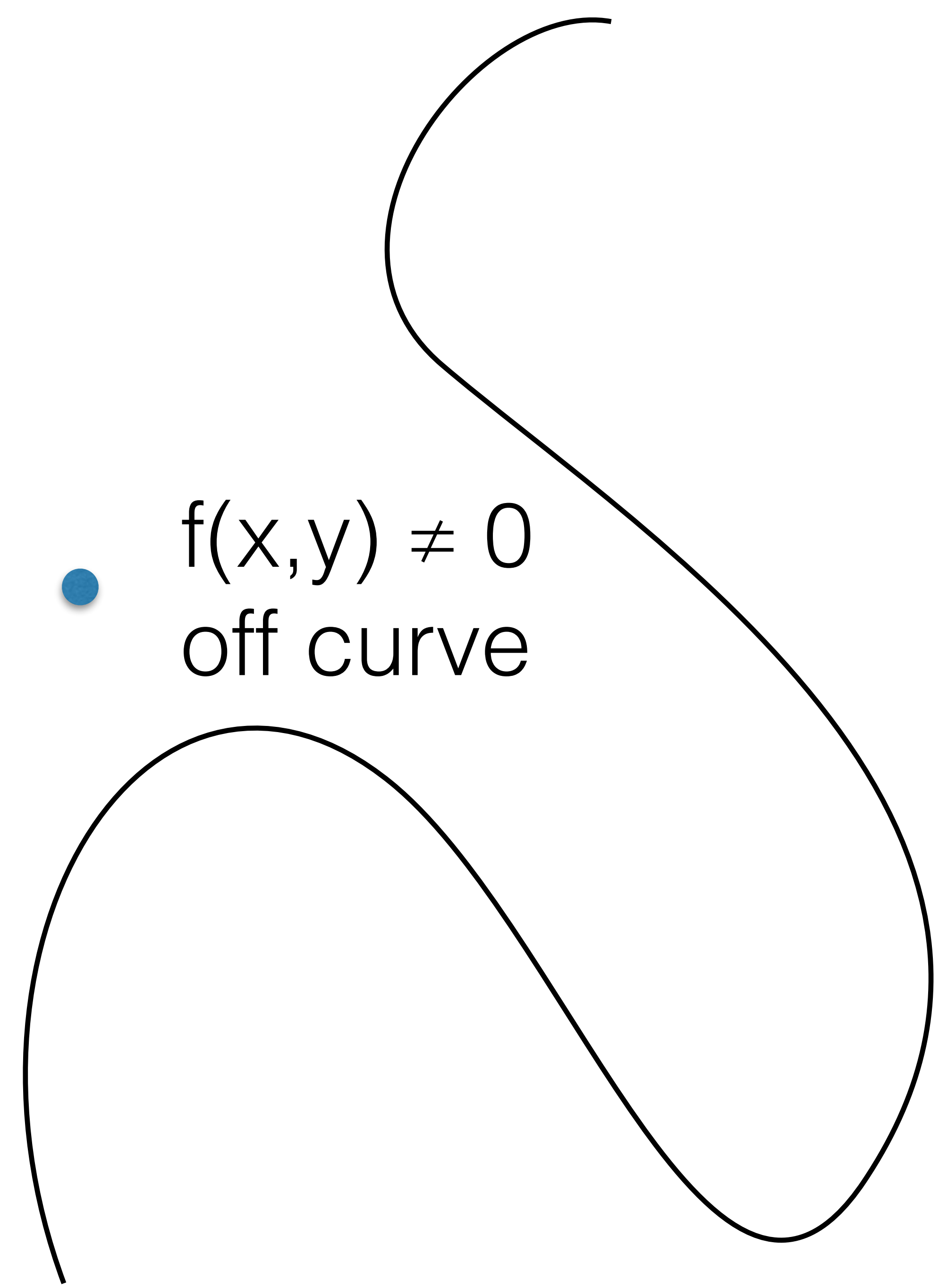

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(2D) $f(x,y) = 0$

test if $(x,y)$ is on the curve

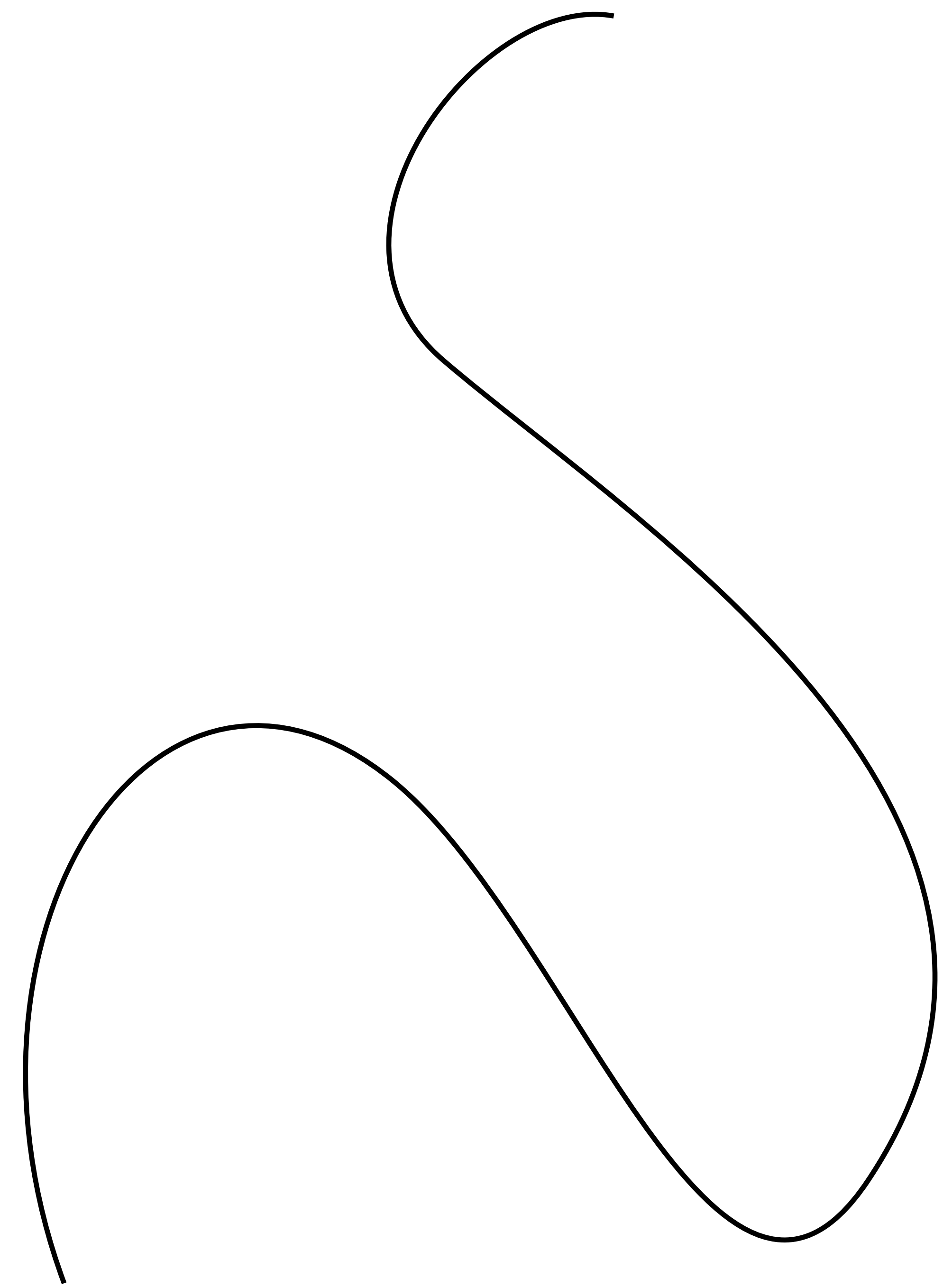
*Parametric*

(2D) $(x,y) = \mathbf{f}(t)$

(3D) $(x,y,z) = \mathbf{f}(t)$

map free *parameter* $t$

to points on the curve



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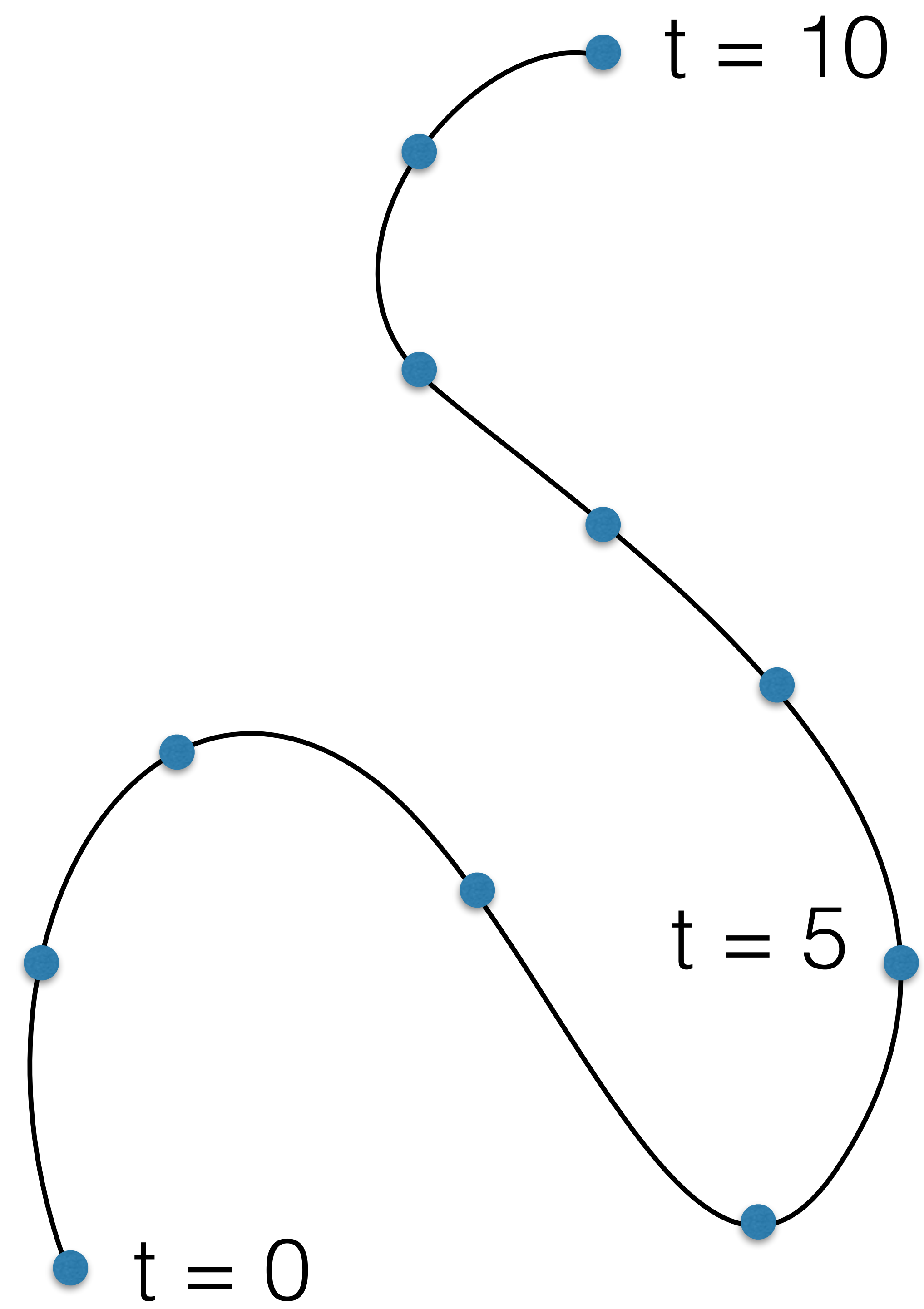


# How do we specify a curve?

*Implicit*

(2D) f(x,y) = 0

test if (x,y) is on the curve

*Parametric*

(2D) (x,y) = **f**(t)

(3D) (x,y,z) = **f**(t)

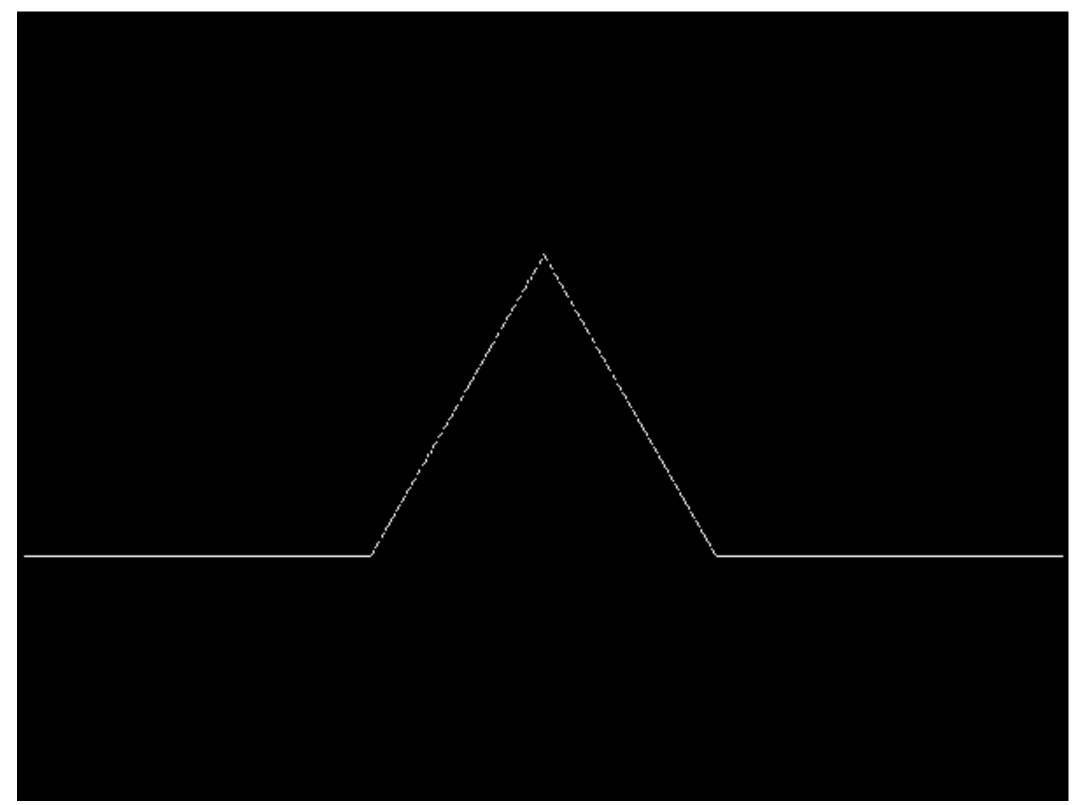
map free *parameter* t

to points on the curve

*Procedural*

e.g., fractals,

subdivision schemes



**Fractal: Koch Curve**

[George Reese]

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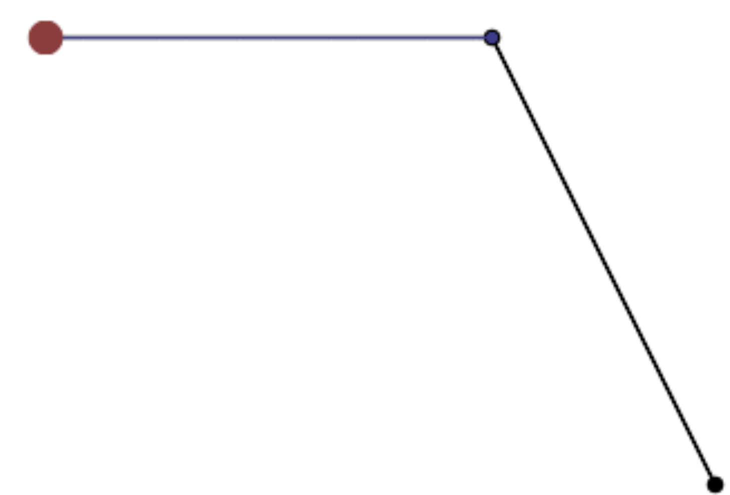
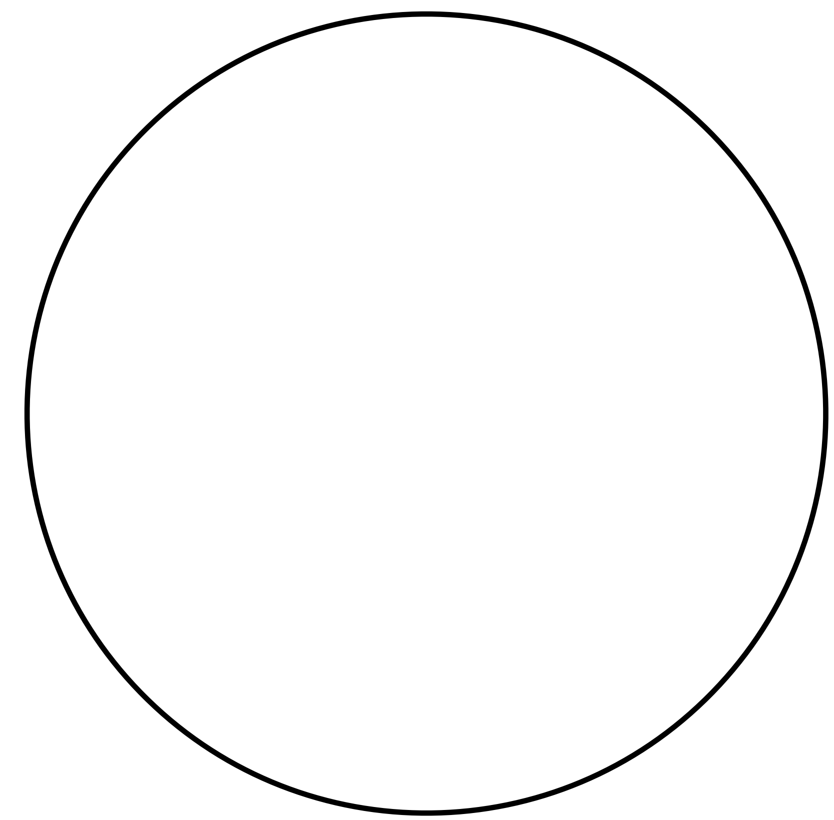


**Bezier Curve**

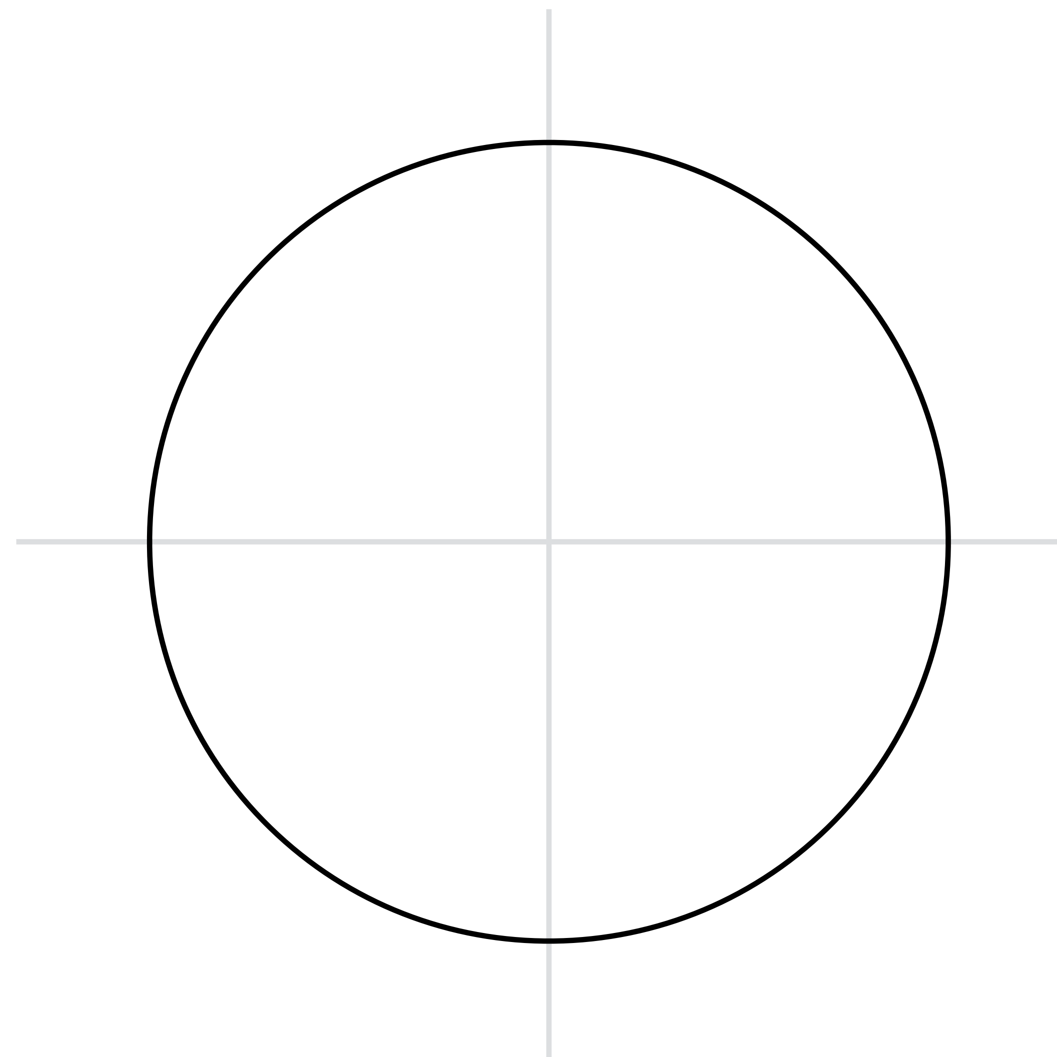
# A curve may have multiple representations



# A curve may have multiple representations

*Implicit*

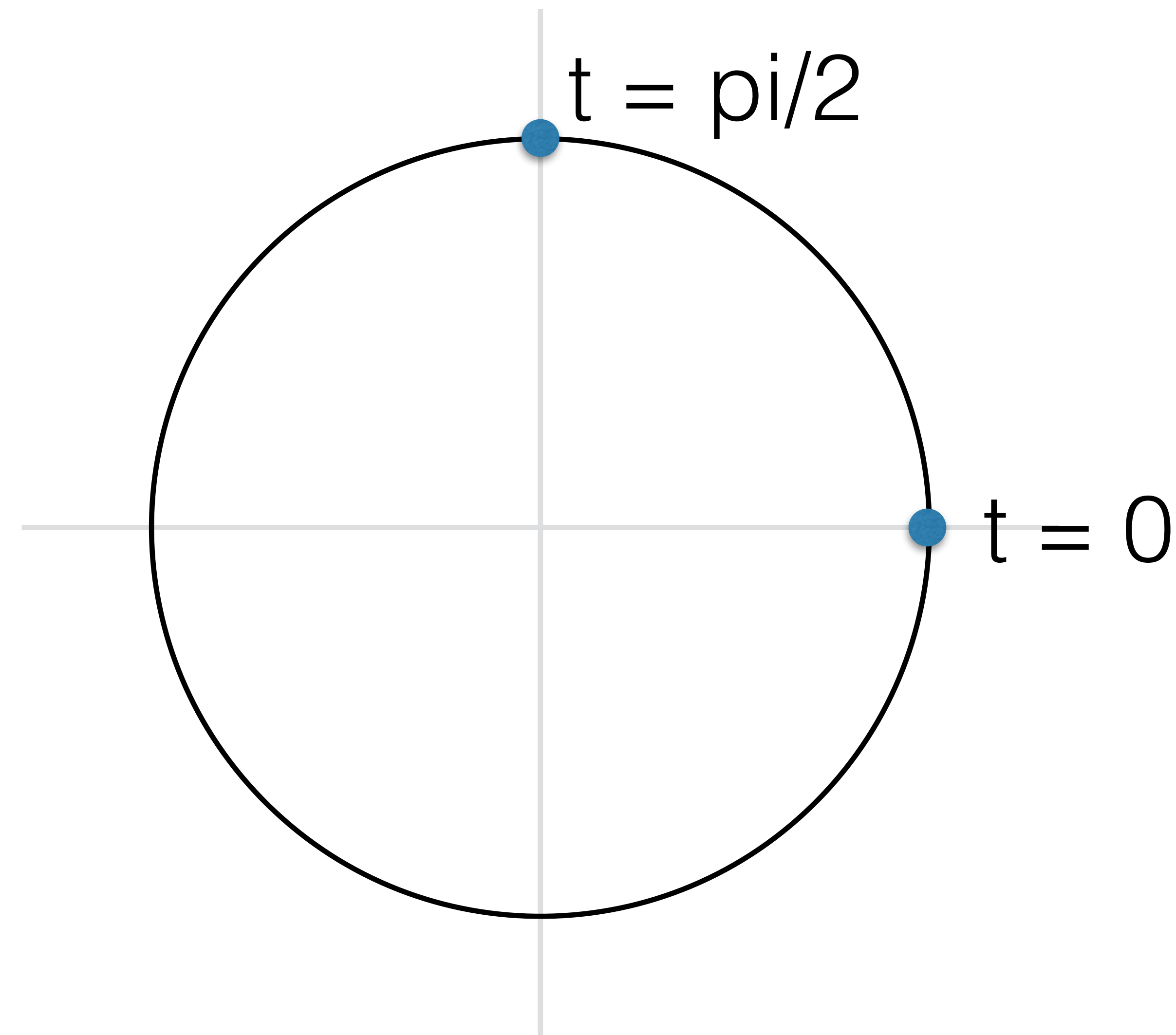
$f(x,y) = x^2 + y^2 - 1 = 0$



# A curve may have multiple representations

*Parametric*

$(x,y) = \mathbf{f}(t) = (\cos t, \sin t)$



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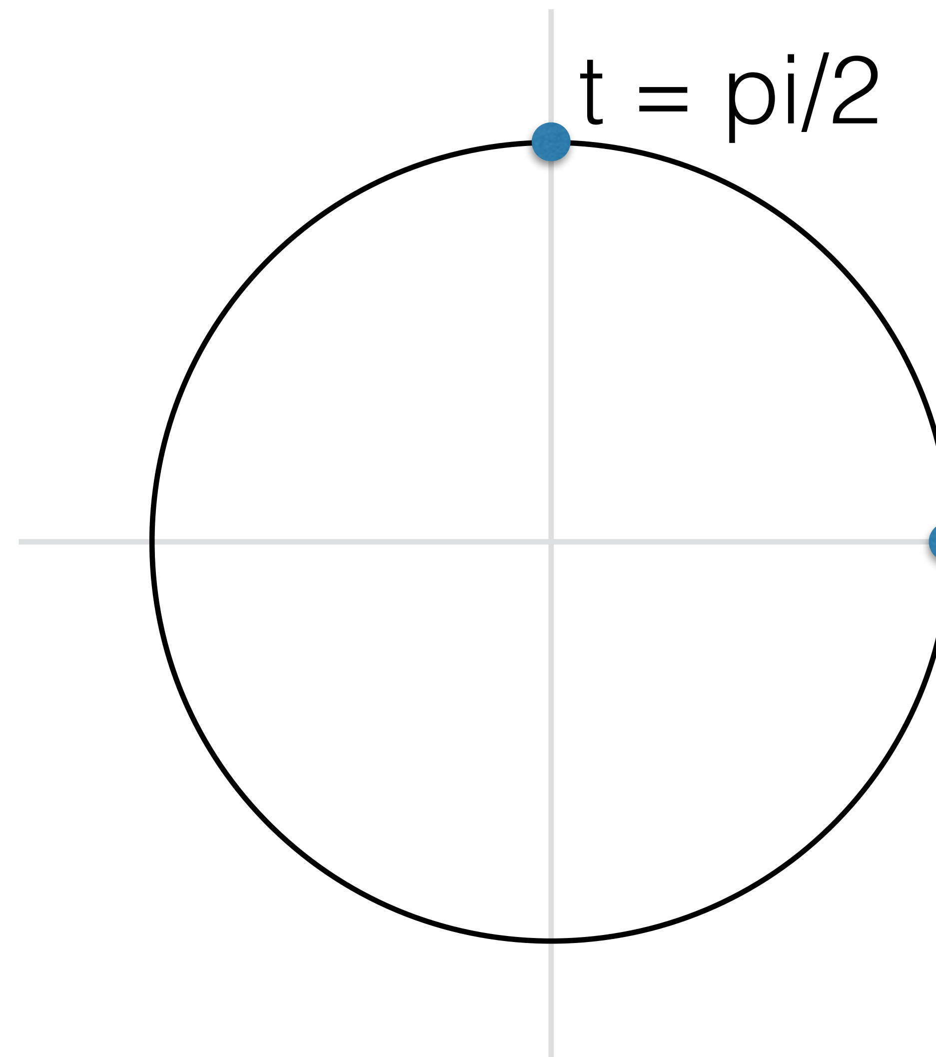
*Parametric*

$(x,y) = \mathbf{f}(t) = (\cos t, \sin t)$,

$t$ in $[0,2\pi)$

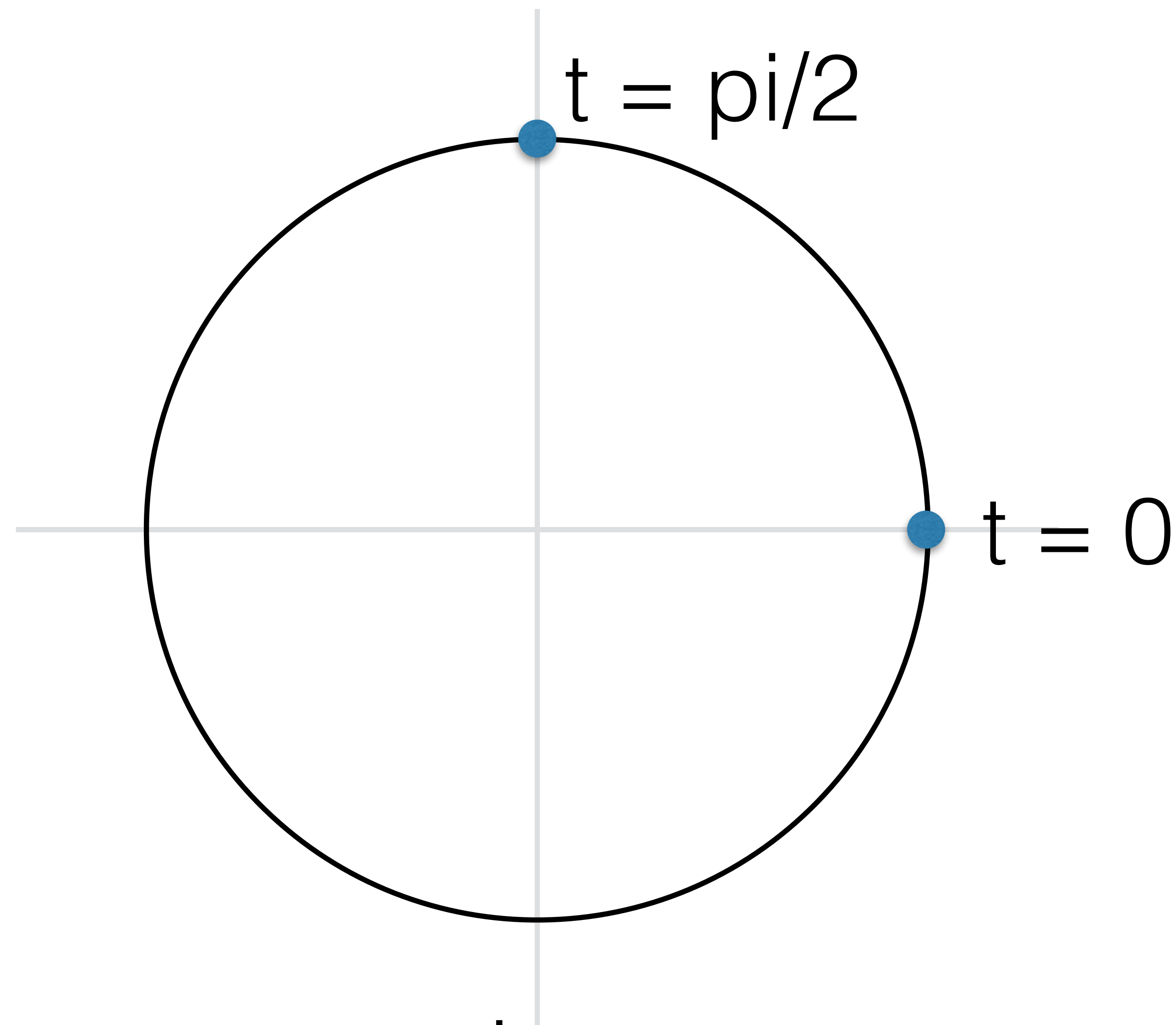



Same curve (set of points),
but different mathematical representation!

# A curve may have multiple representations

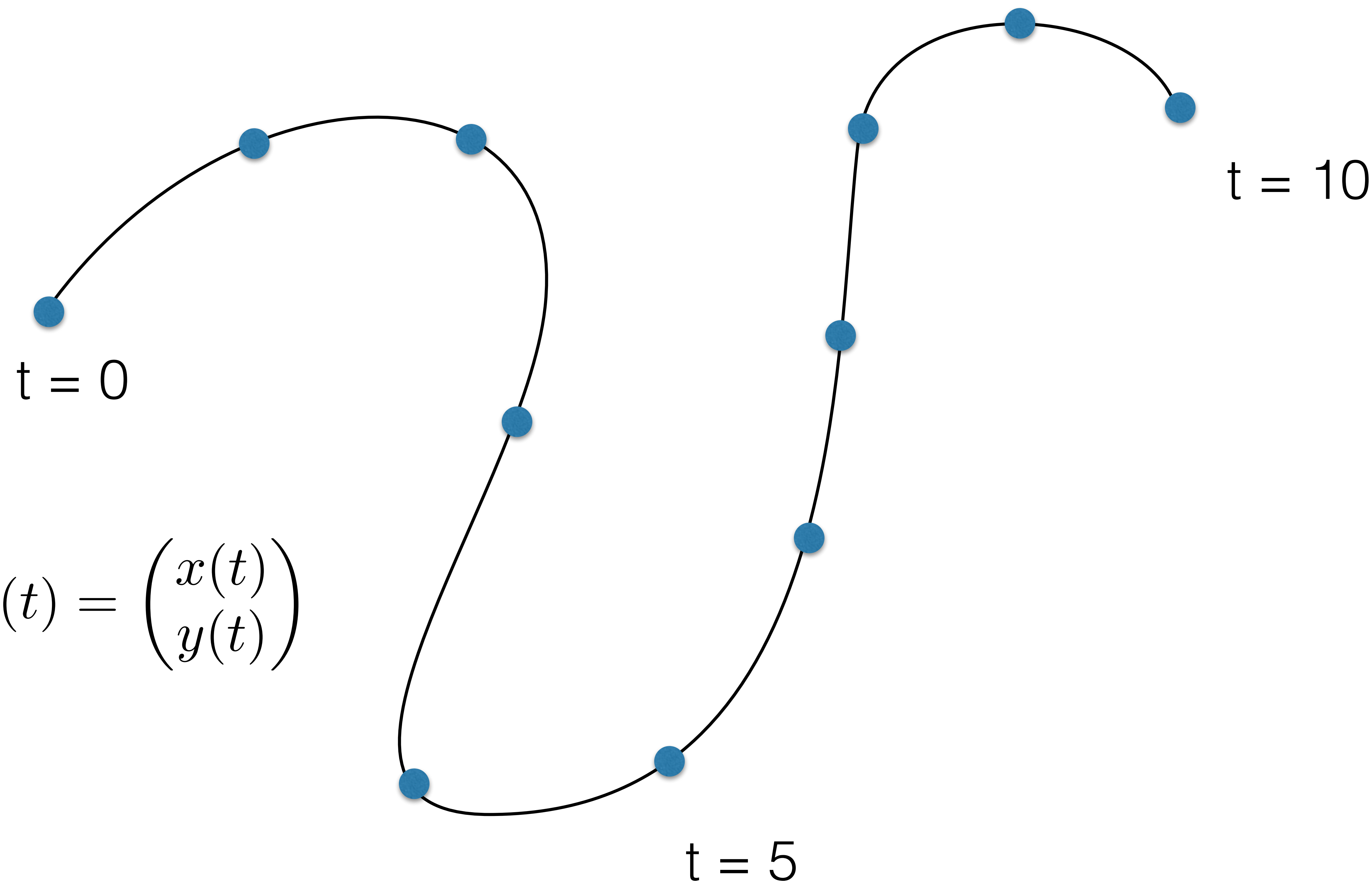
*Parametric*

(x,y) = **f**(t) = (cos t, sin t),
t in [0,2pi)



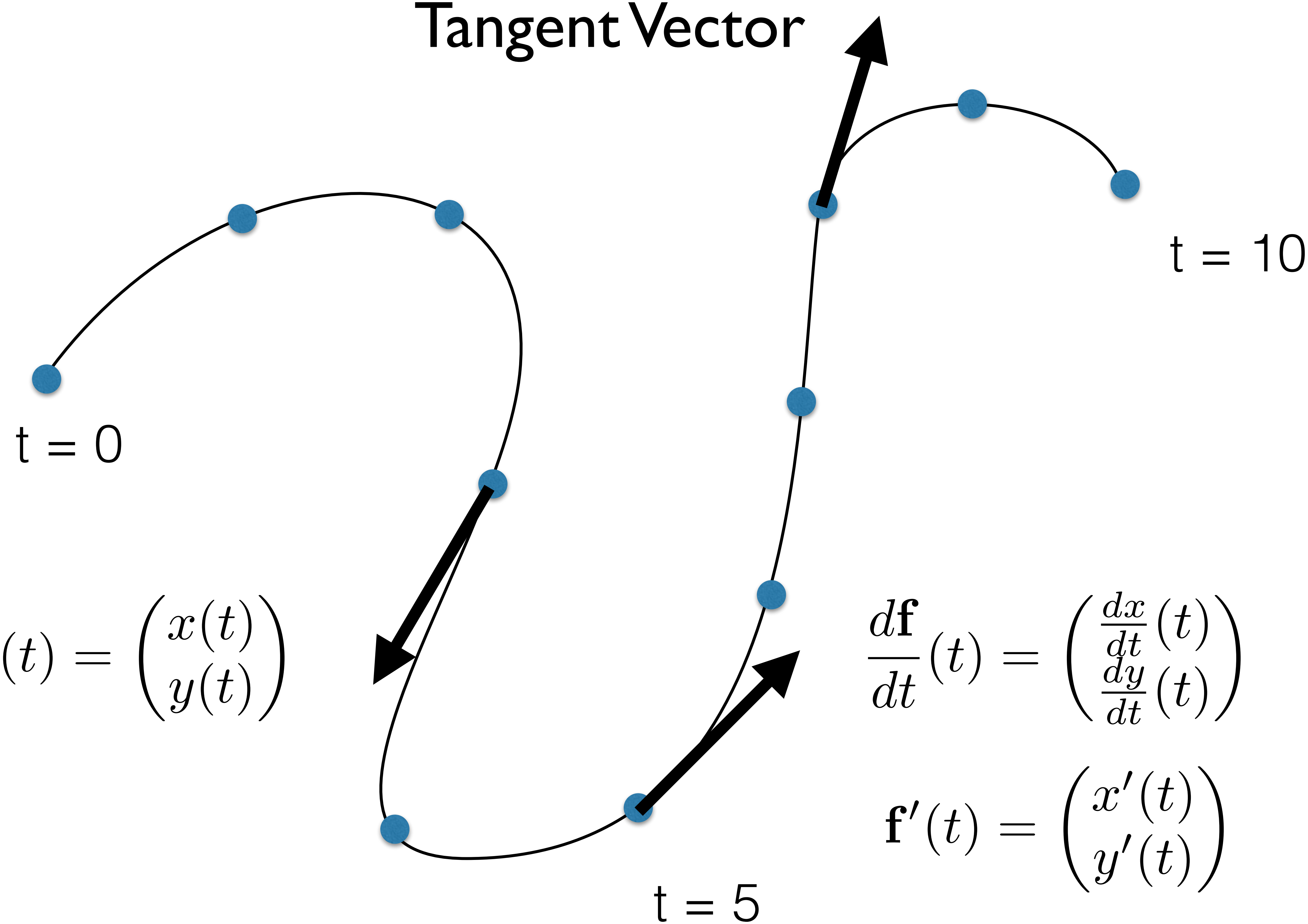

We will focus on parametric representations

# Parametric Form




$$\mathbf{f}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

# Parametric Form
# Tangent Vector

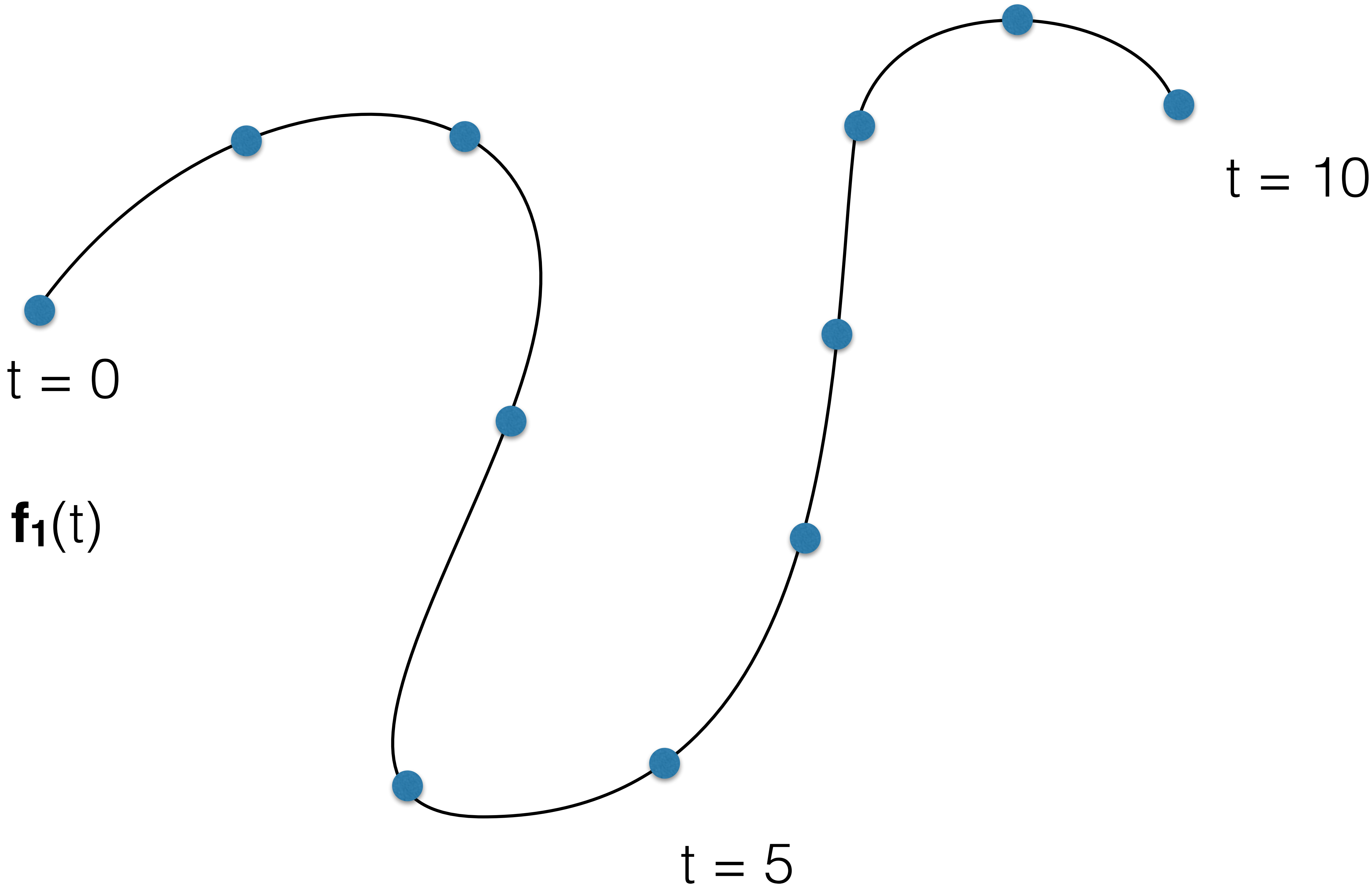



$$\mathbf{f}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\frac{d\mathbf{f}}{dt}(t) = \begin{pmatrix} \frac{dx}{dt}(t) \\ \frac{dy}{dt}(t) \end{pmatrix}$$

$$\mathbf{f}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

# Parameterization, re-parameterization



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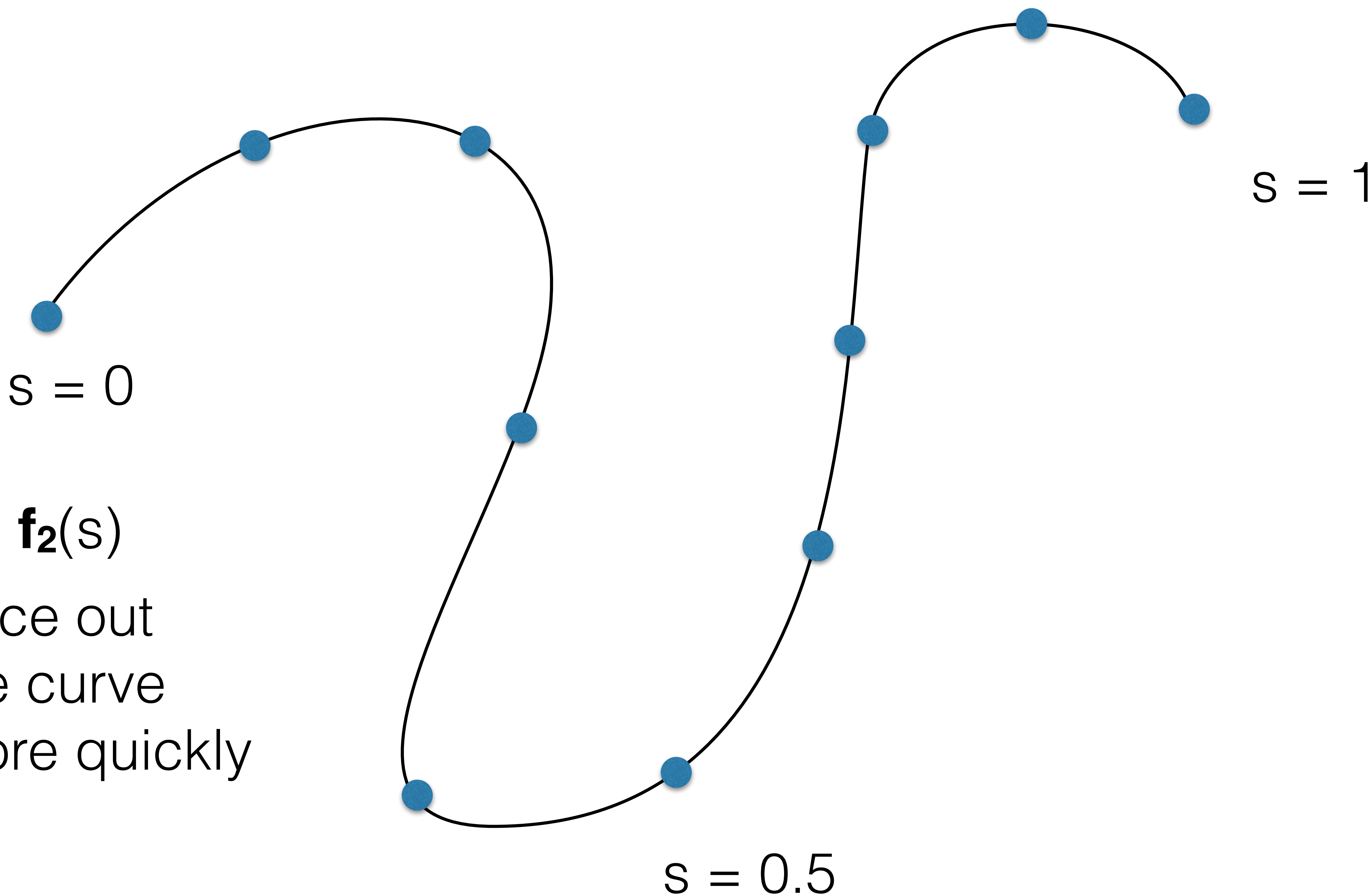
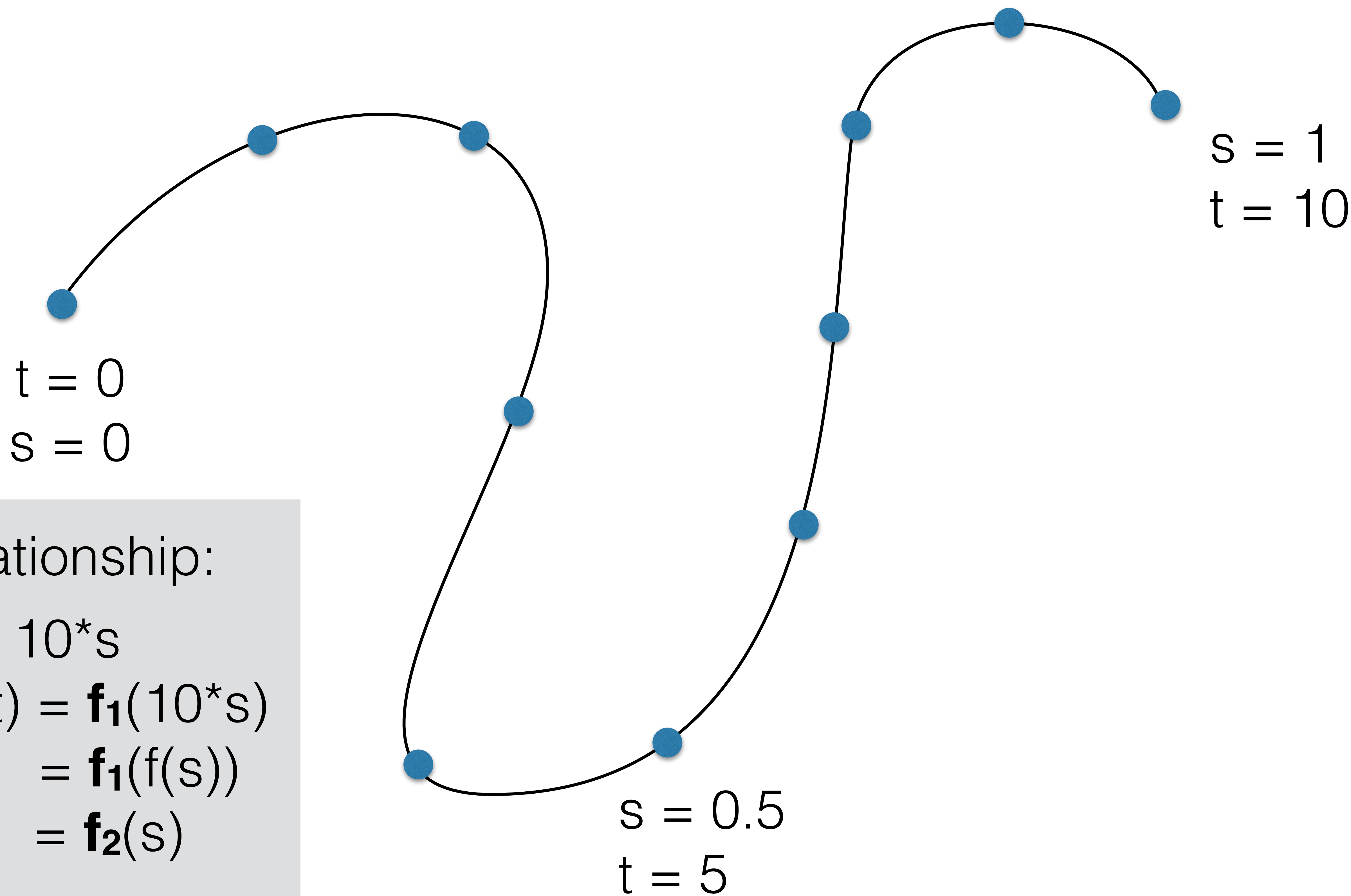



$\mathbf{f_2}(s)$

trace out
the curve
more quickly

# Parameterization, re-parameterization




relationship:

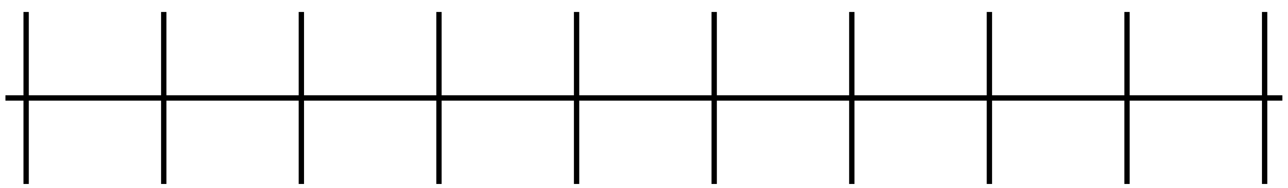
$t = 10*s$

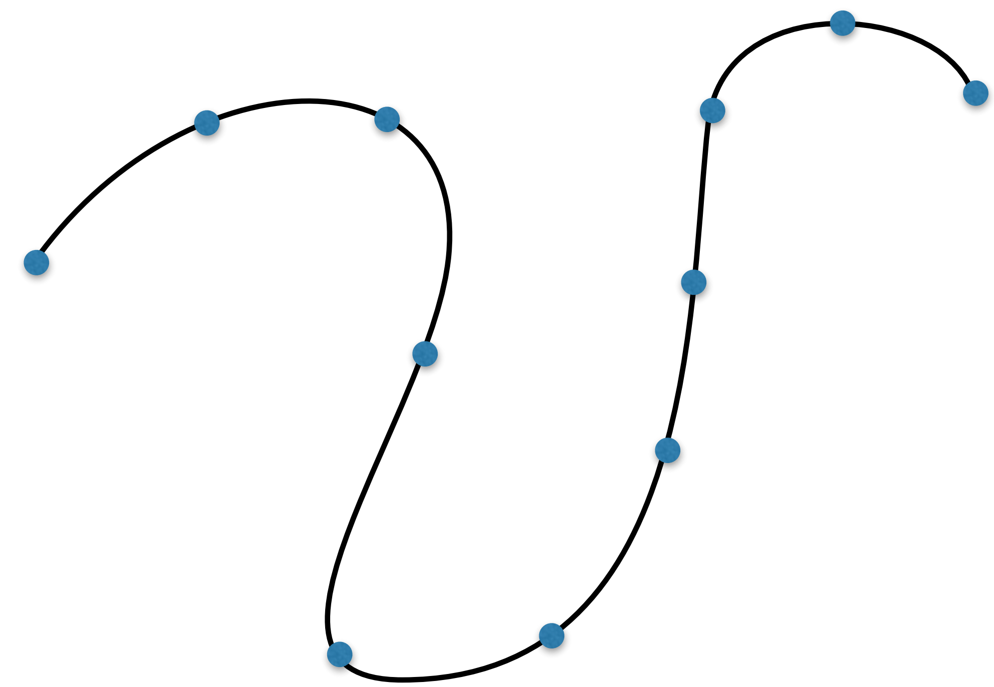
$\mathbf{f_1}(t) = \mathbf{f_1}(10*s)$

$= \mathbf{f_1}(f(s))$

$= \mathbf{f_2}(s)$

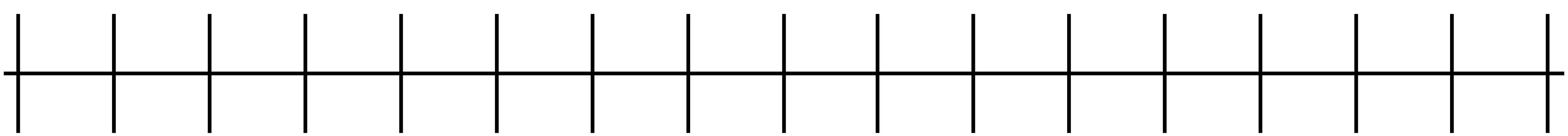
# Parameterization, re-parameterization





$t = 0$ $t = 10$

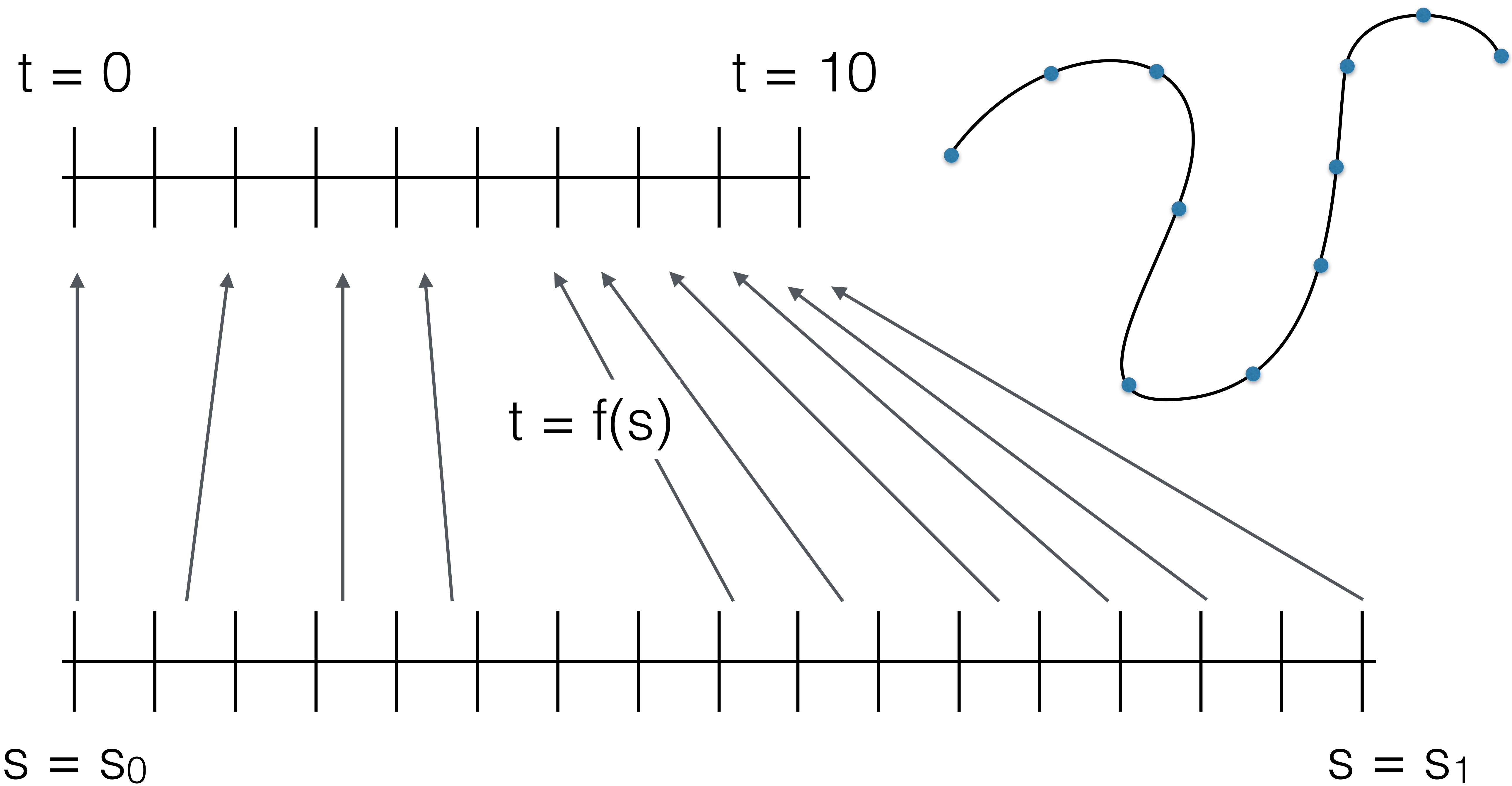
$\mathbf{f_1}(t)$



$s = s_0$ $s = s_1$

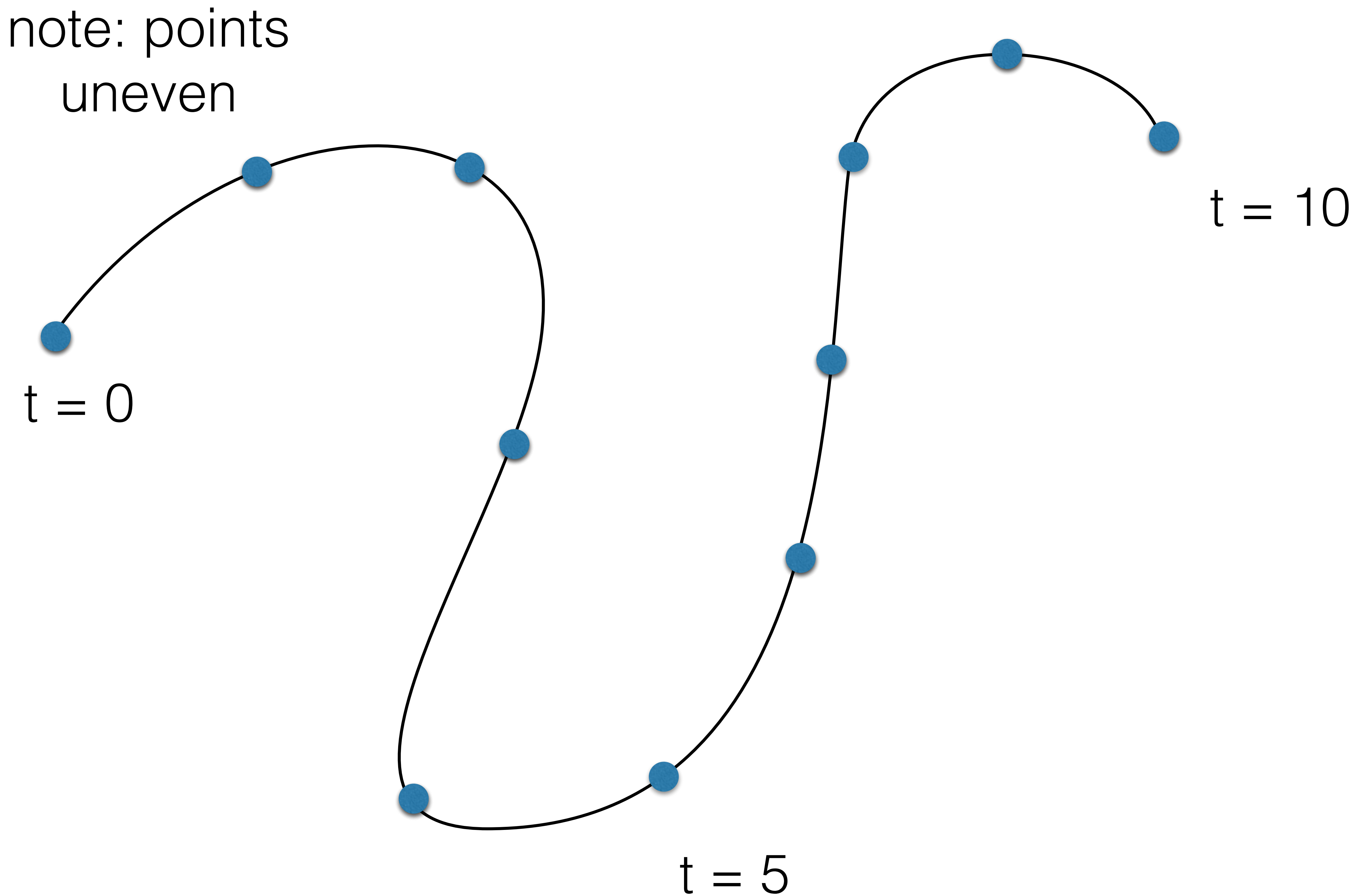
$\mathbf{f_2}(s) = \mathbf{f_1}(f(s))$

# Parameterization, re-parameterization



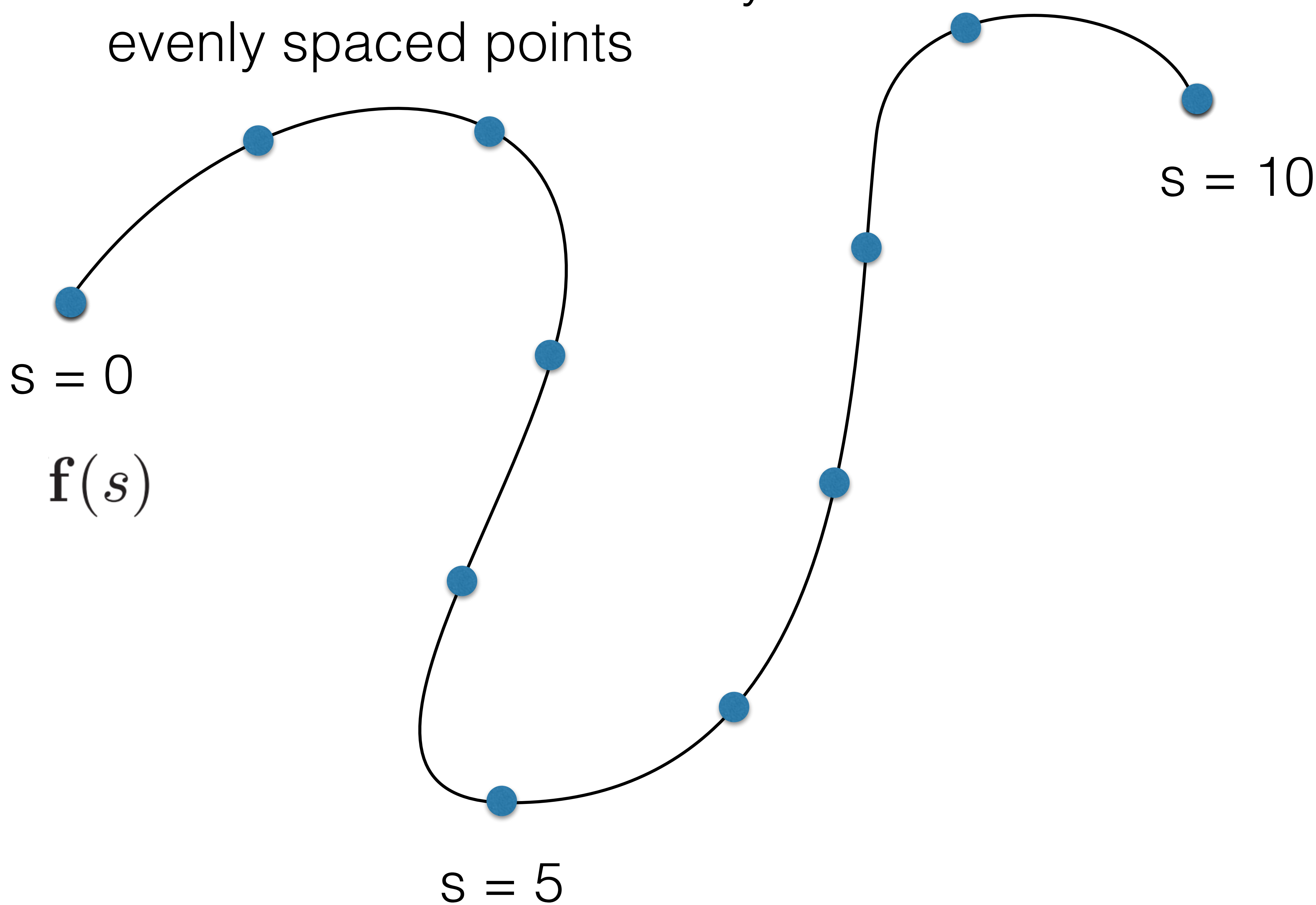

$$\mathbf{f_2}(s) = \mathbf{f_1}(f(s))$$

# Natural parameterization



# Natural parameterization

pen moves at a constant velocity:
evenly spaced points



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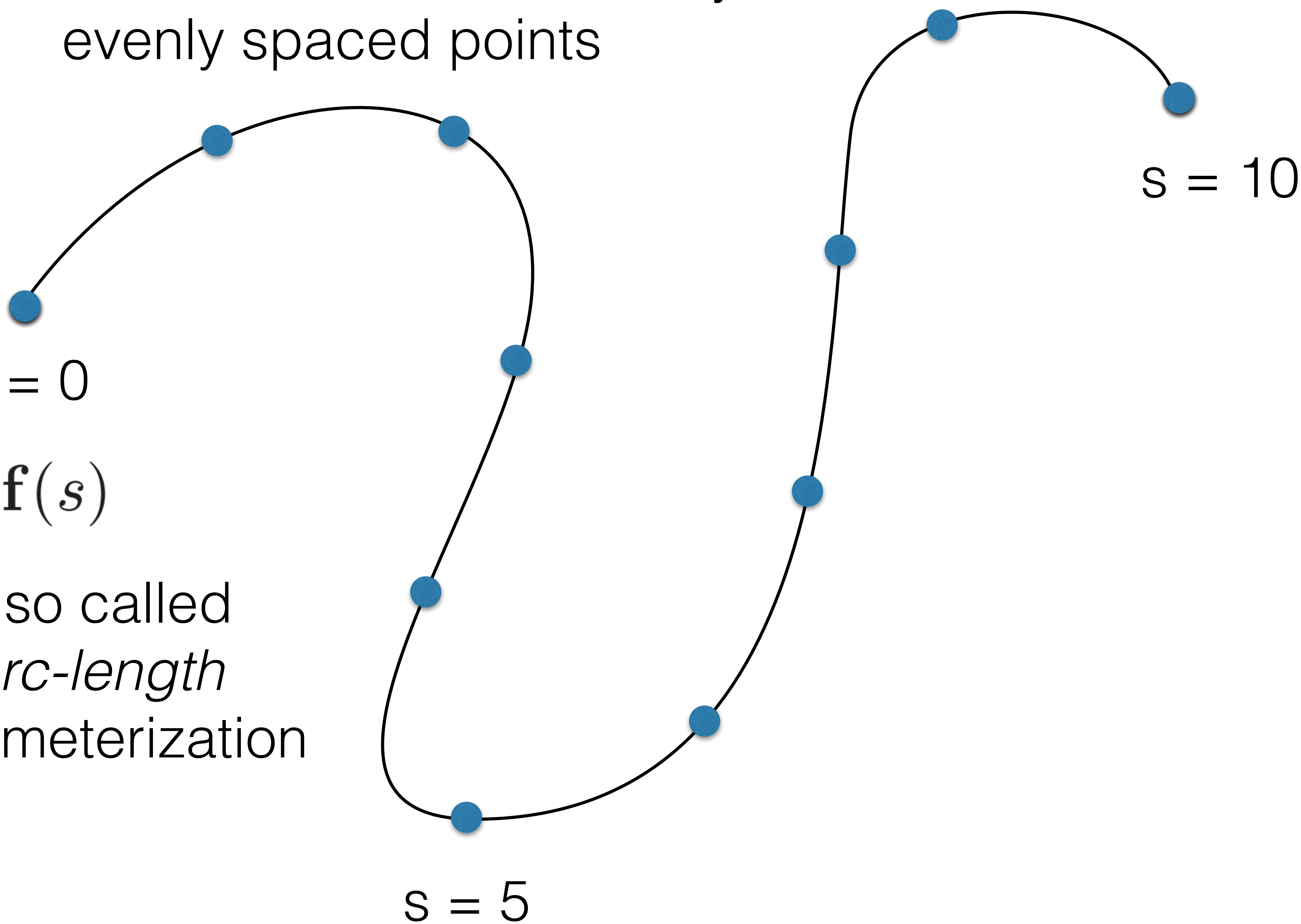


$\mathbf{f}(s)$

also called
*arc-length*
parameterization

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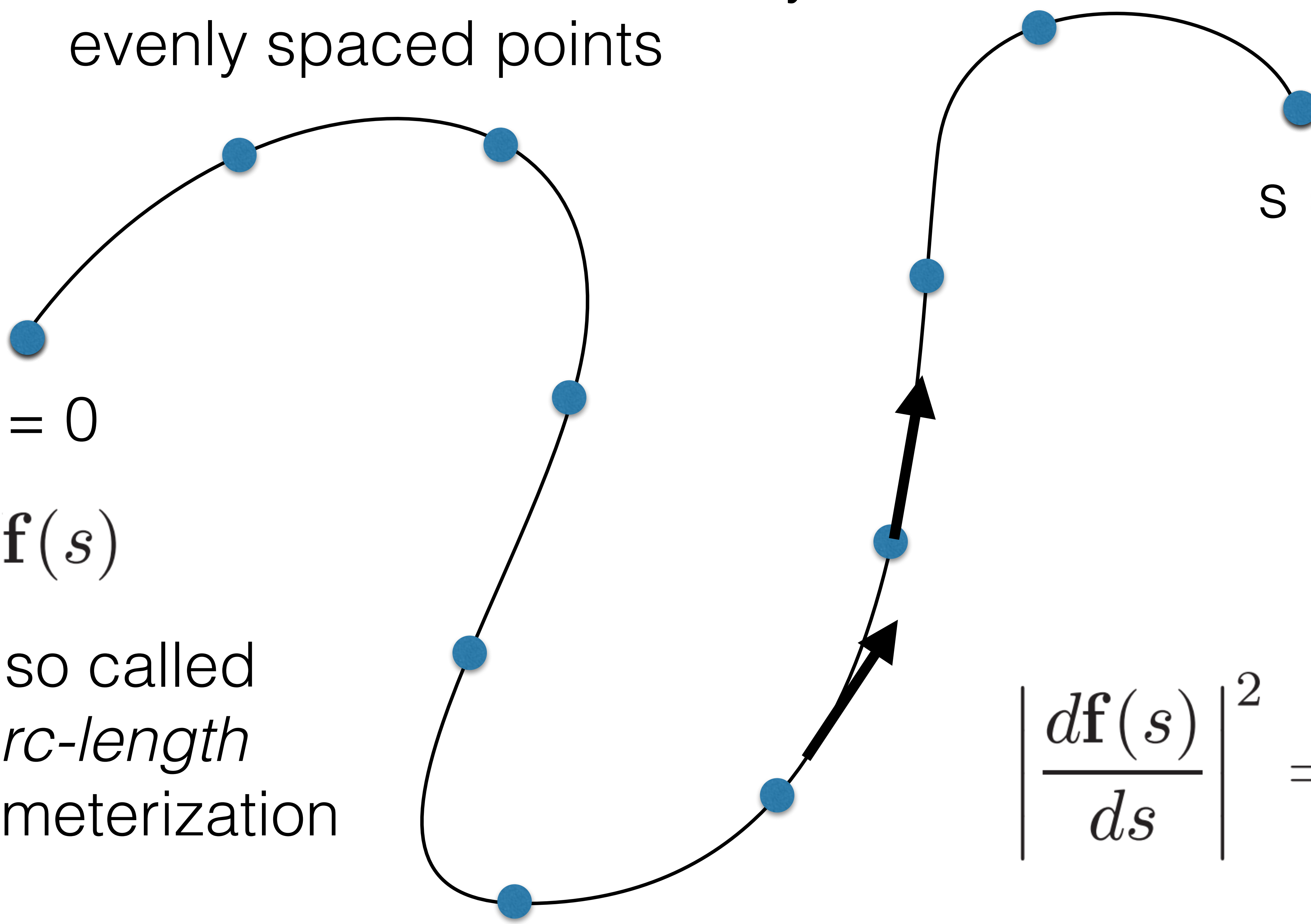



$$\mathbf{f}(s)$$

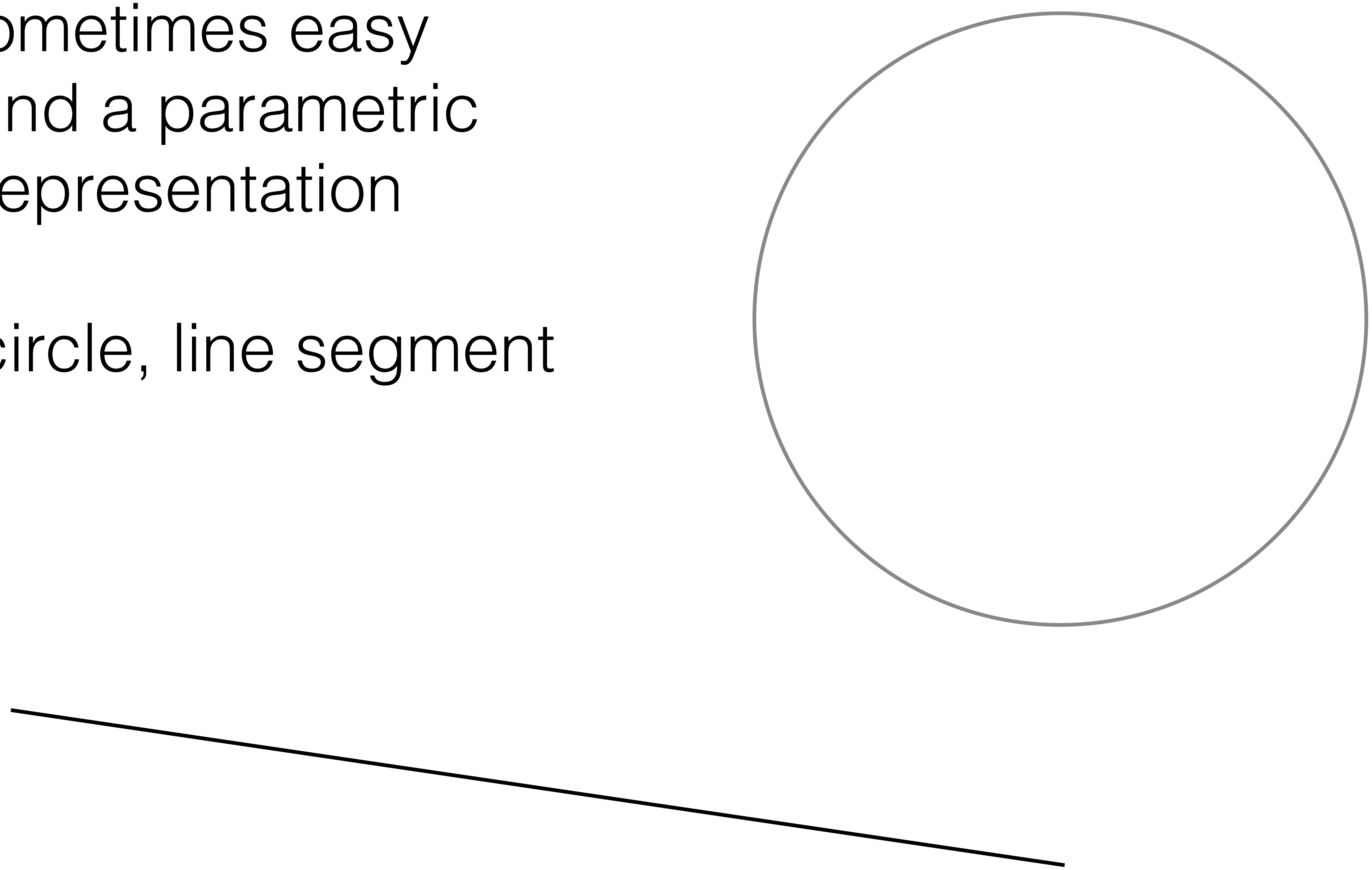
also called
*arc-length*
parameterization

$$\left|\frac{d\mathbf{f}(s)}{ds}\right|^2 = c$$

# piecewise parametric representation

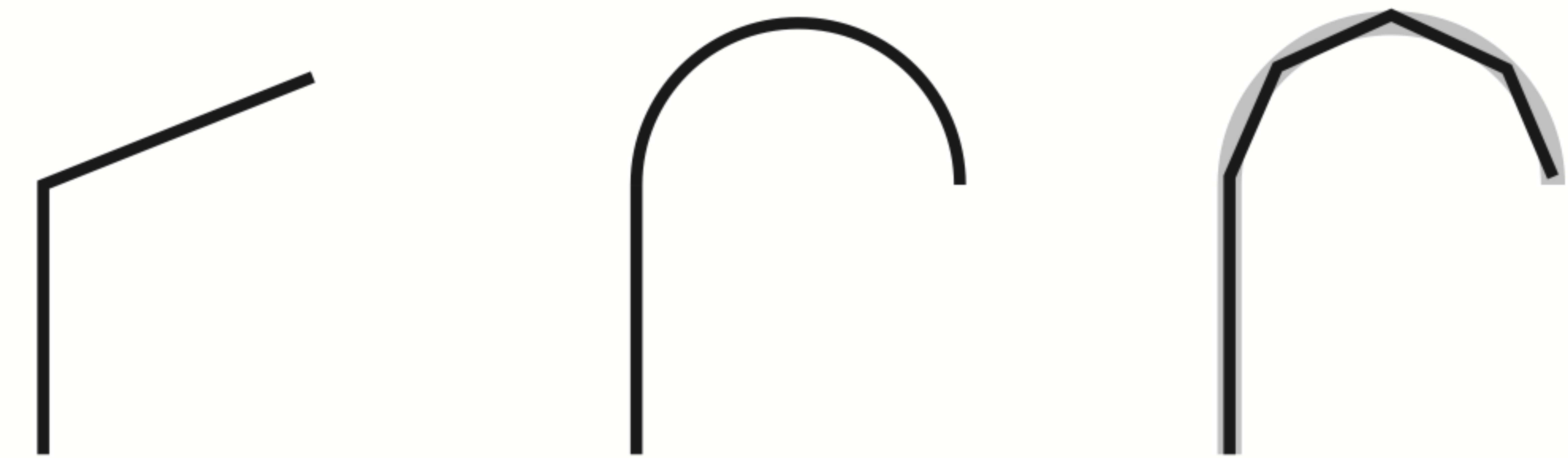
sometimes easy
to find a parametric
representation

e.g., circle, line segment



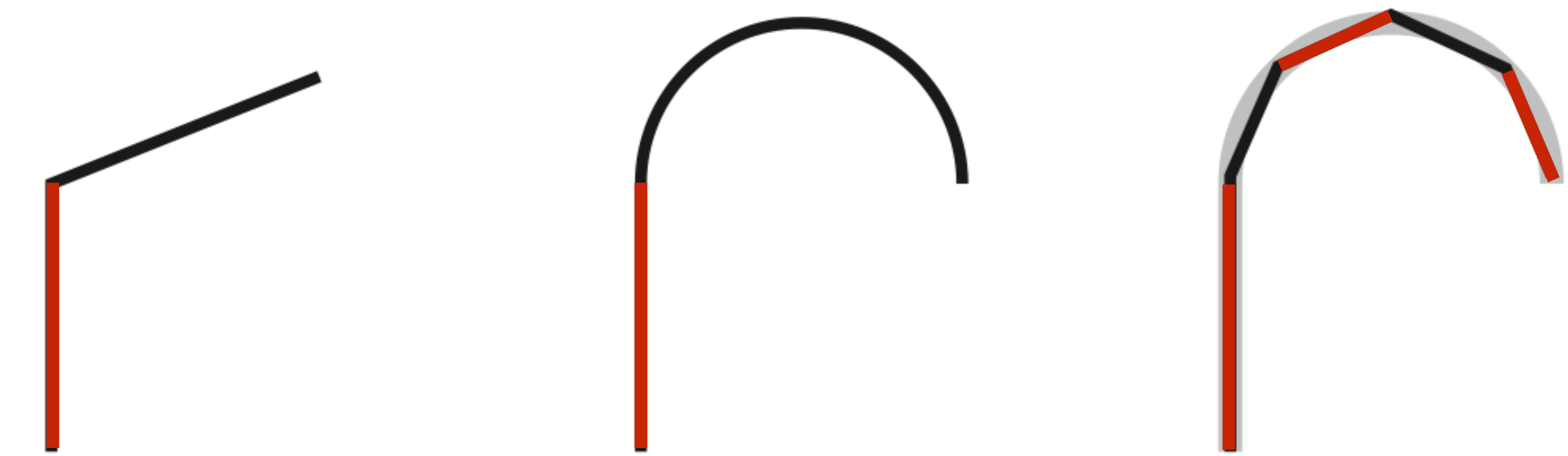
# piecewise parametric representation

in other cases, not obvious



# piecewise parametric representation

strategy: break into simpler pieces



# piecewise parametric representation

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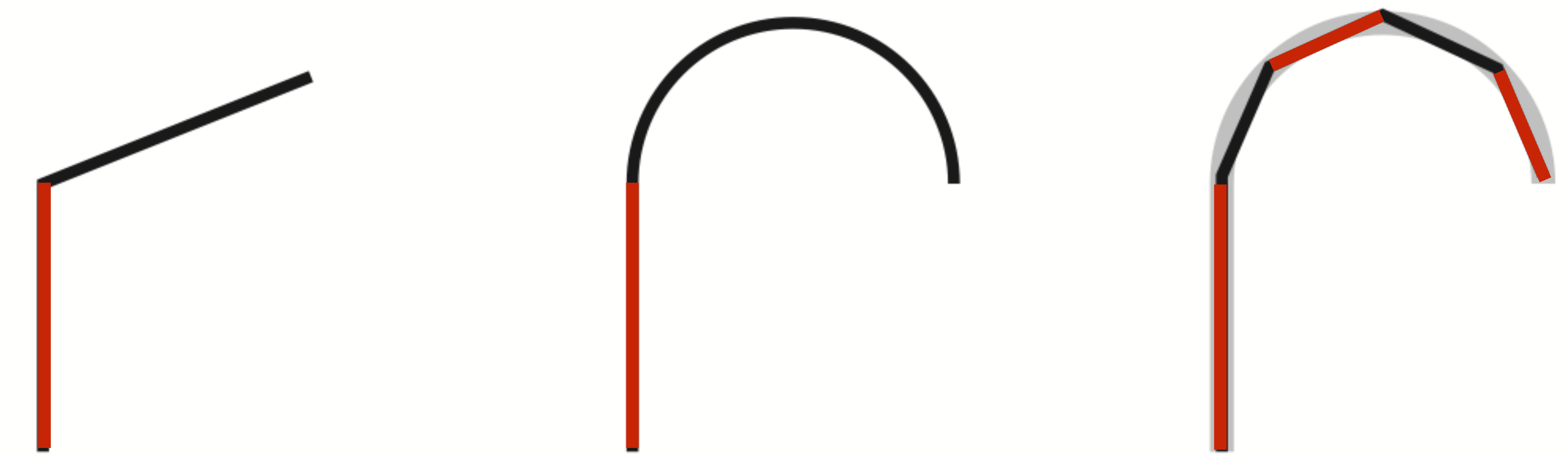

switch between functions that represent pieces:

$$\mathbf{f}(u) = \begin{cases} \mathbf{f}_1(2u) & u \leq 0.5 \\ \mathbf{f}_2(2u-1) & u > 0.5 \end{cases}$$

# piecewise parametric representation

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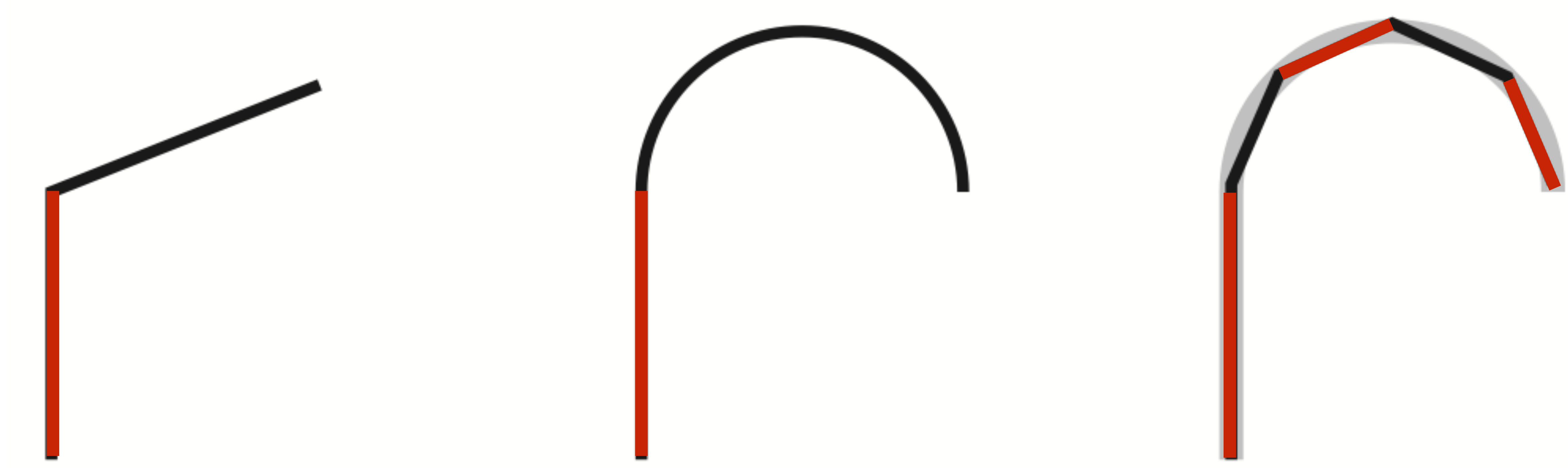

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map the inputs to **f**$_1$ and **f**$_2$ to be from 0 to 1

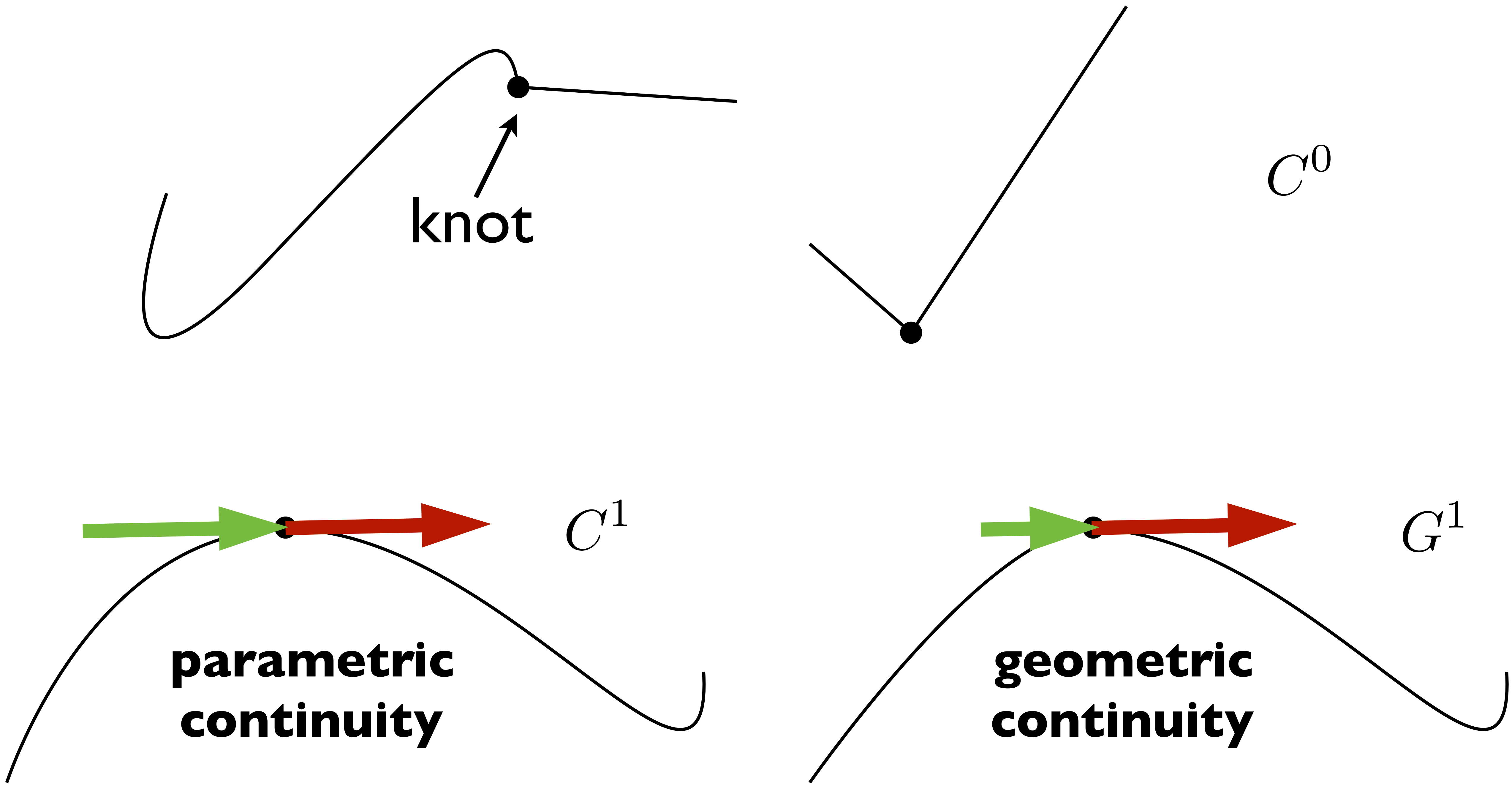
# Curve Properties

Local properties:
- continuity
- position
- direction
- curvature

Global properties (examples):
- closed curve
- curve crosses itself

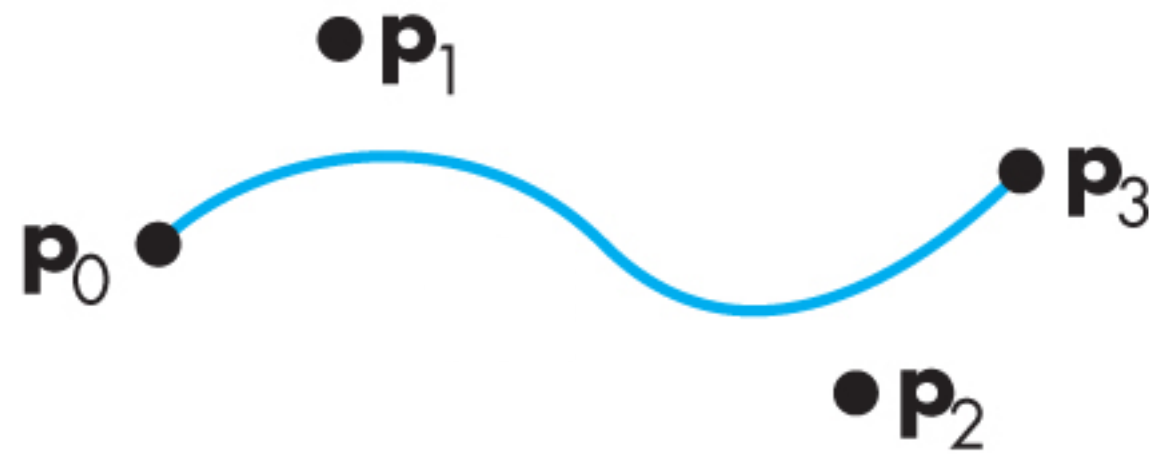
Interpolating vs. non-interpolating

# Continuity: stitching curve segments together



# Interpolating vs. Approximating Curves


**Interpolating**




**Approximating**
(non-interpolating)

# Finding a Parametric Representation

# Polynomial Pieces

$$f(u) = a_0 + a_1 u + a_2 u^2 + \cdots + a_n u^n$$

# Polynomial Pieces

**coefficients** **n = degree**

...

$$f(u) = a_0 + a_1 u + a_2 u^2 + \cdots + a_n u^n$$

# Polynomial Pieces

**coefficients** ... **n = degree**

$$f(u) = a_0 + a_1 u + a_2 u^2 + \cdots + a_n u^n$$

"canonical form" (monomial basis)

# *Blending functions* are more convenient basis than monomial basis


- "canonical form" (monomial basis)

$$\mathbf{f}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3$$

- "geometric form" (blending functions)

$$\mathbf{f}(u) = b_0(u)\mathbf{p}_0 + b_1(u)\mathbf{p}_1 + b_2(u)\mathbf{p}_2 + b_3(u)\mathbf{p}_3$$

# *Blending functions* are more convenient basis than monomial basis

$$f(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix} \qquad \mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$f(u) = \mathbf{u} \cdot \mathbf{a} = \mathbf{u}^T \mathbf{a}$$

# *Blending functions* are more convenient basis than monomial basis

$$C\mathbf{a} = \mathbf{p}$$

$$\mathbf{a} = C^{-1}\mathbf{p} = B\mathbf{p}$$

$$\mathbf{p} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$\begin{aligned} f(u) = \mathbf{u}^T\mathbf{a} &= \mathbf{u}^T(B\mathbf{p}) \\ &= (\mathbf{u}^T B)\mathbf{p} \\ &= \mathbf{b}(u)^T\mathbf{p} \end{aligned}$$

$$\mathbf{b}(u) = \begin{pmatrix} b_0(u) \\ b_1(u) \\ b_2(u) \\ b_3(u) \end{pmatrix}$$

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$$C\mathbf{a} = \mathbf{p}$$

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$$f(u) = \mathbf{u}^T\mathbf{a} = \mathbf{u}^T(B\mathbf{p})$$

$$= (\mathbf{u}^T B)\mathbf{p}$$

$$= \mathbf{b}(u)^T\mathbf{p}$$

$$\mathbf{b}(u) = \begin{pmatrix} \vdots \\ b_3(u) \end{pmatrix}$$

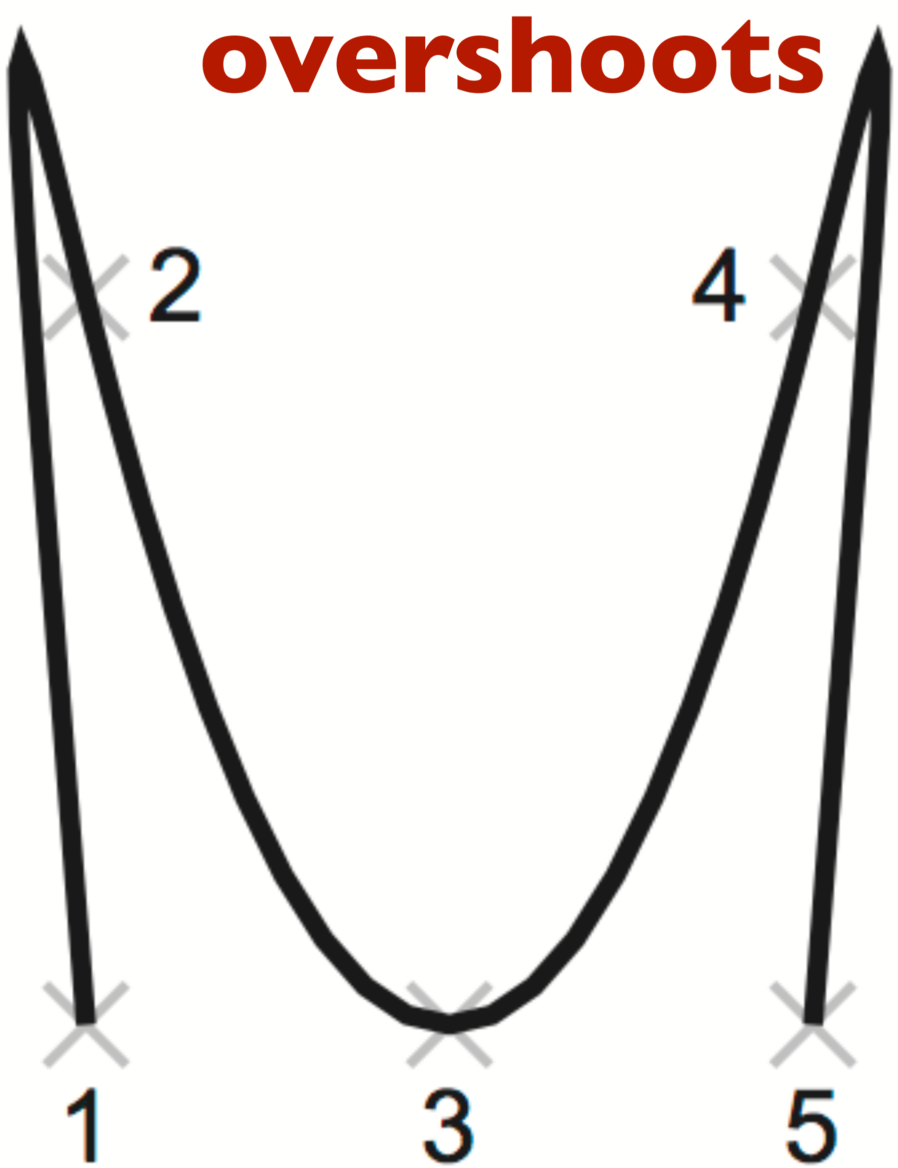
**Some examples <whiteboard>**

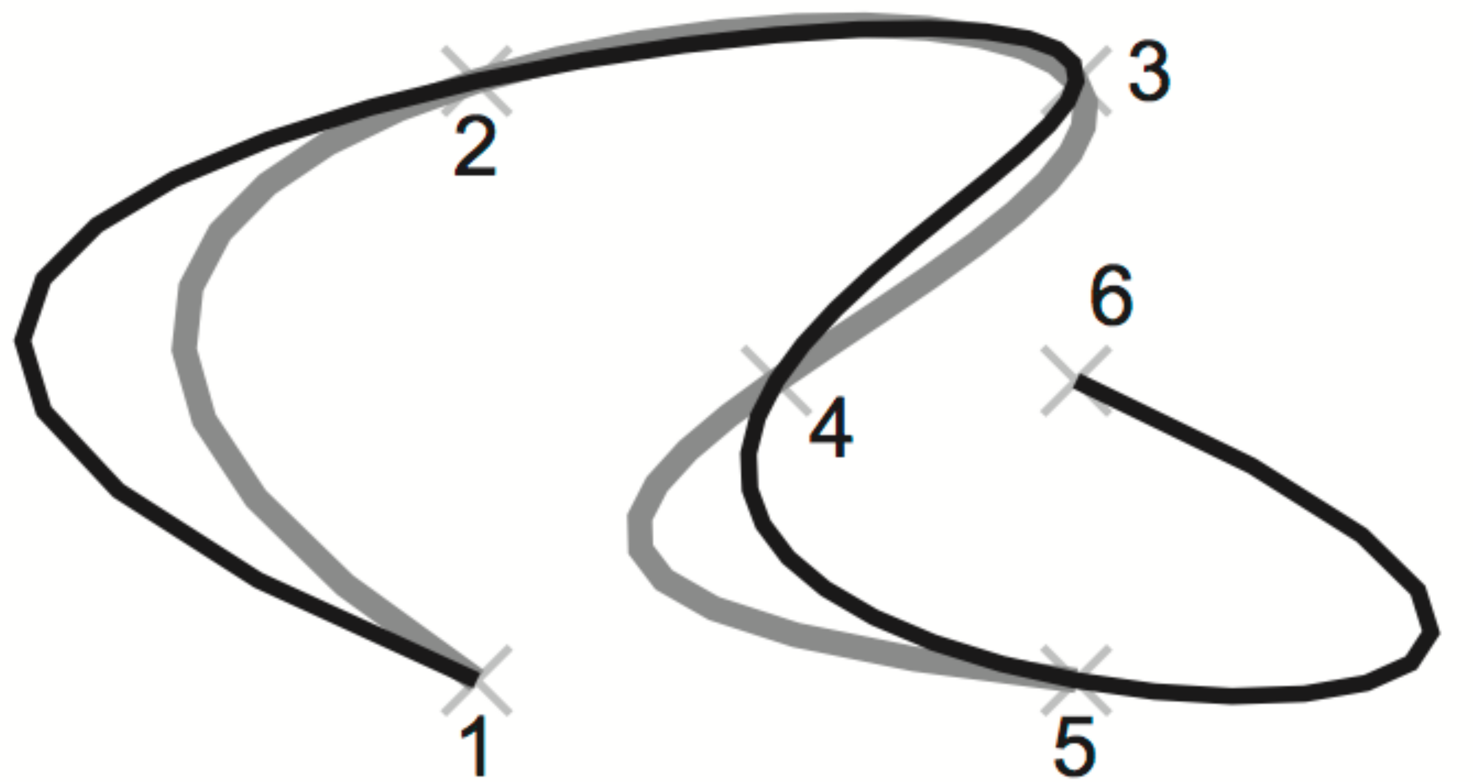
# Interpolating Polynomials

# Interpolating polynomials

- Given n+1 data points, can find a unique interpolating polynomial of degree n
- Different methods:
  - Vandermonde matrix
  - Lagrange interpolation
  - Newton interpolation

# higher order interpolating polynomials are rarely used



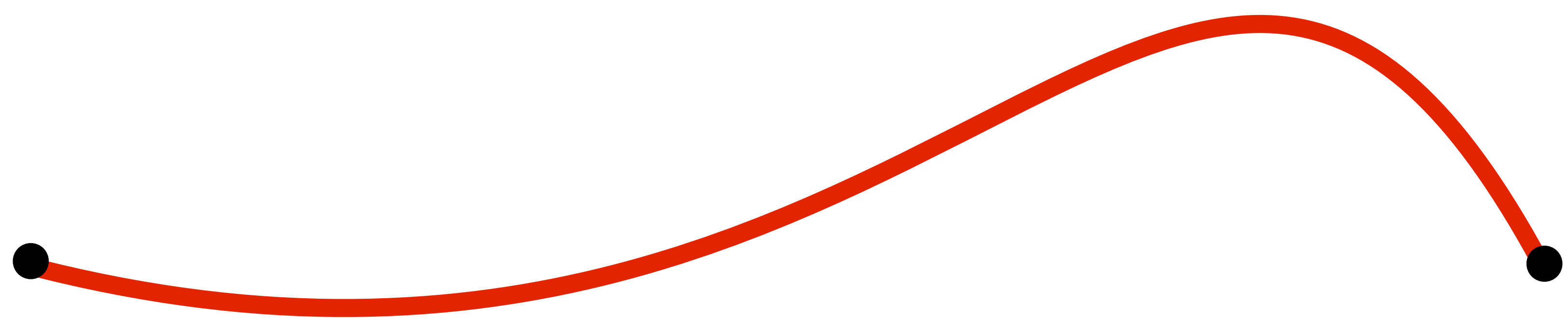




**non-local effects**

4th order (gray) to 5th order (black)

# Piecewise Polynomial Curves

# Cubics



$$\mathbf{f}(u) = \mathbf{a}_0 + \mathbf{a}_1 u + \mathbf{a}_2 u^2 + \mathbf{a}_3 u^3$$

- Allow up to $C^2$ continuity at knots
- need 4 control points
  - may be 4 points on the curve, combination of points and derivatives, ...
- good smoothness and computational properties

# Advantages of Cubics

- allow for C2 continuity (C1 often not enough, more than C2 unnecessary)
- n piecewise cubics for n+3 points give minimum curvature curve
- symmetry: position and derivatives can be specified at beginning and end
- good tradeoff between numerical issues and smoothness

# We can get any 3 of 4 properties

1. piecewise cubic
2. curve interpolates control points
3. curve has local control
4. curves has C2 continuity at knots

# Natural Cubics

- C2 continuity
- n points -> n-1 cubic segments
- control is non-local :(
- ill-conditioned x(
- properties 1, 2, 4 (piecewise cubic, curve interpolates control points, curves has C2 continuity at knots)

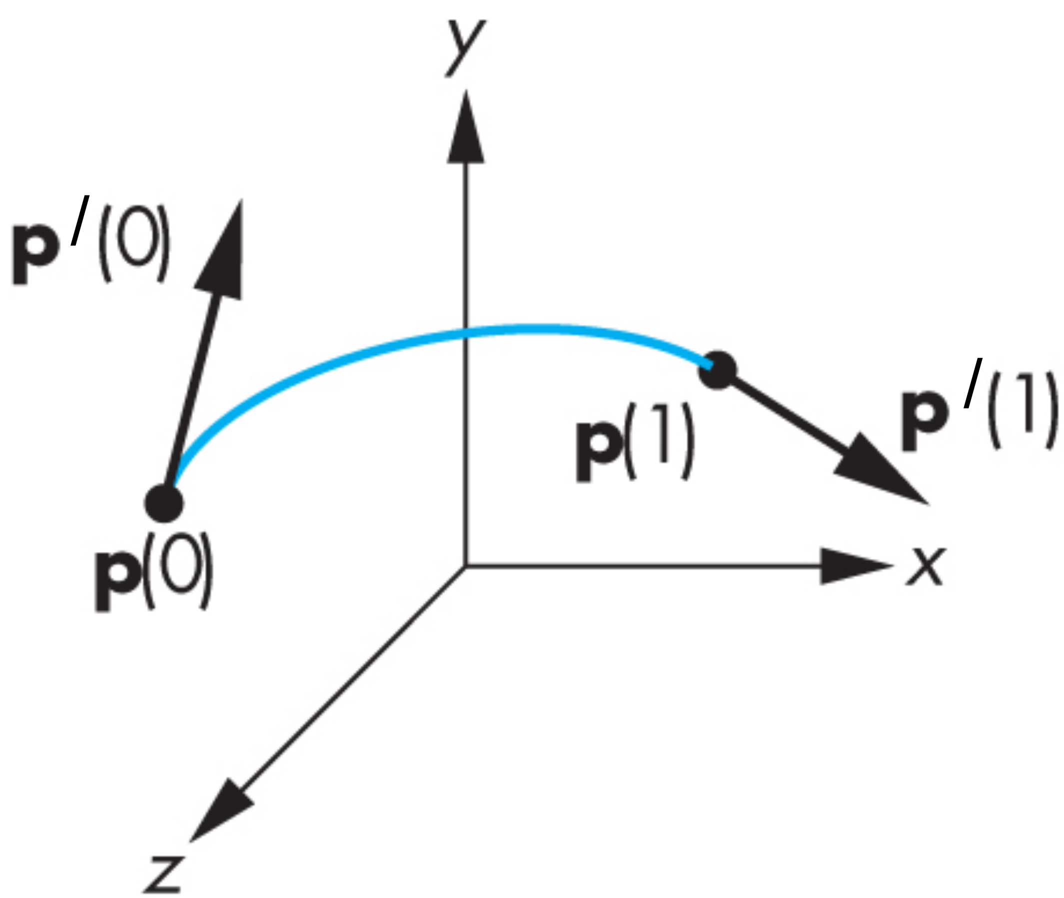
# Cubic Hermite Curves

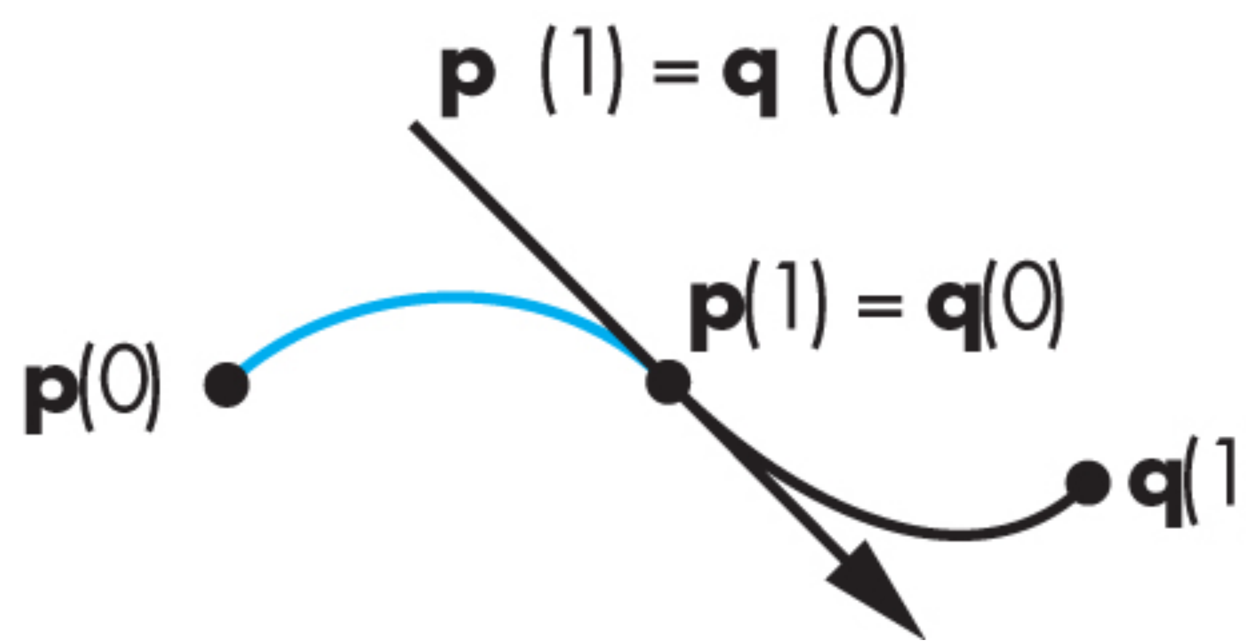
- C1 continuity
- specify both positions and derivatives
- properties 1, 2, 3 (piecewise cubic, curve interpolates control points, curve has local control)

# Cubic Hermite Curves

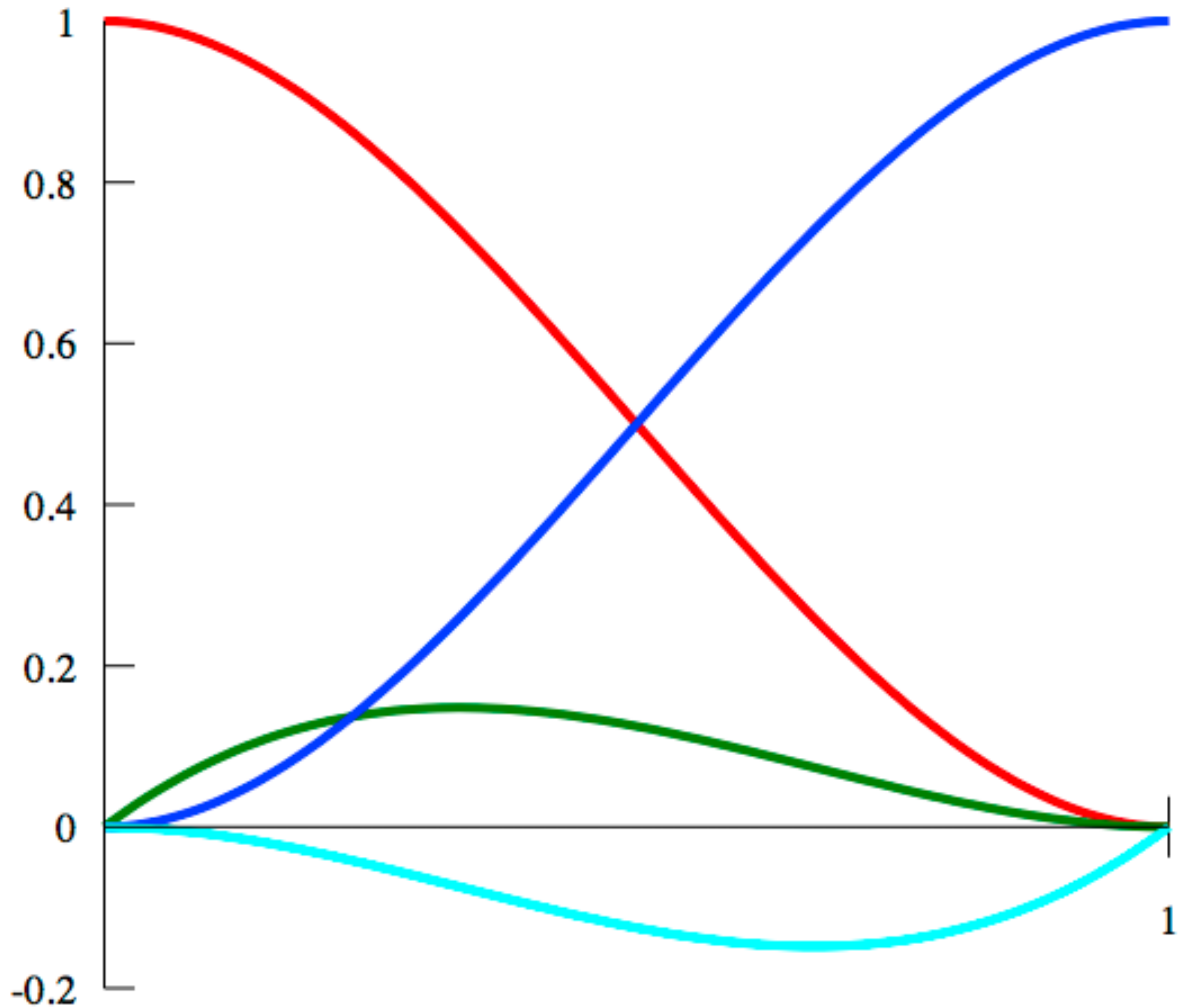
Specify endpoints and derivatives

construct curve with $C^1$ continuity



# Hermite blending functions




[Wikimedia Commons]

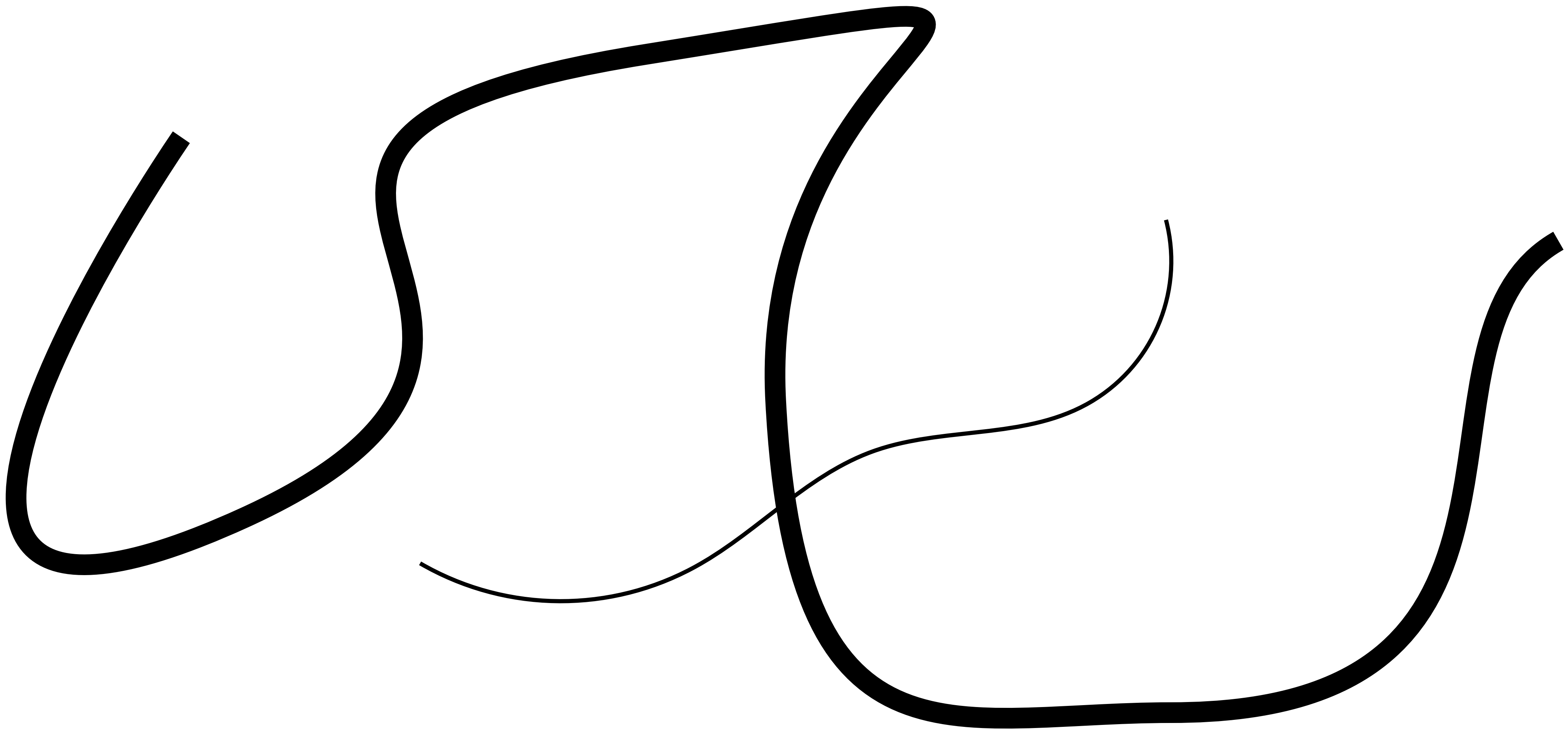
$$b_0(u) = 2u^3 - 3u^2 + 1$$

$$b_1(u) = -2u^3 + 3u^2$$

$$b_2(u) = u^3 - 2u^2 + u$$

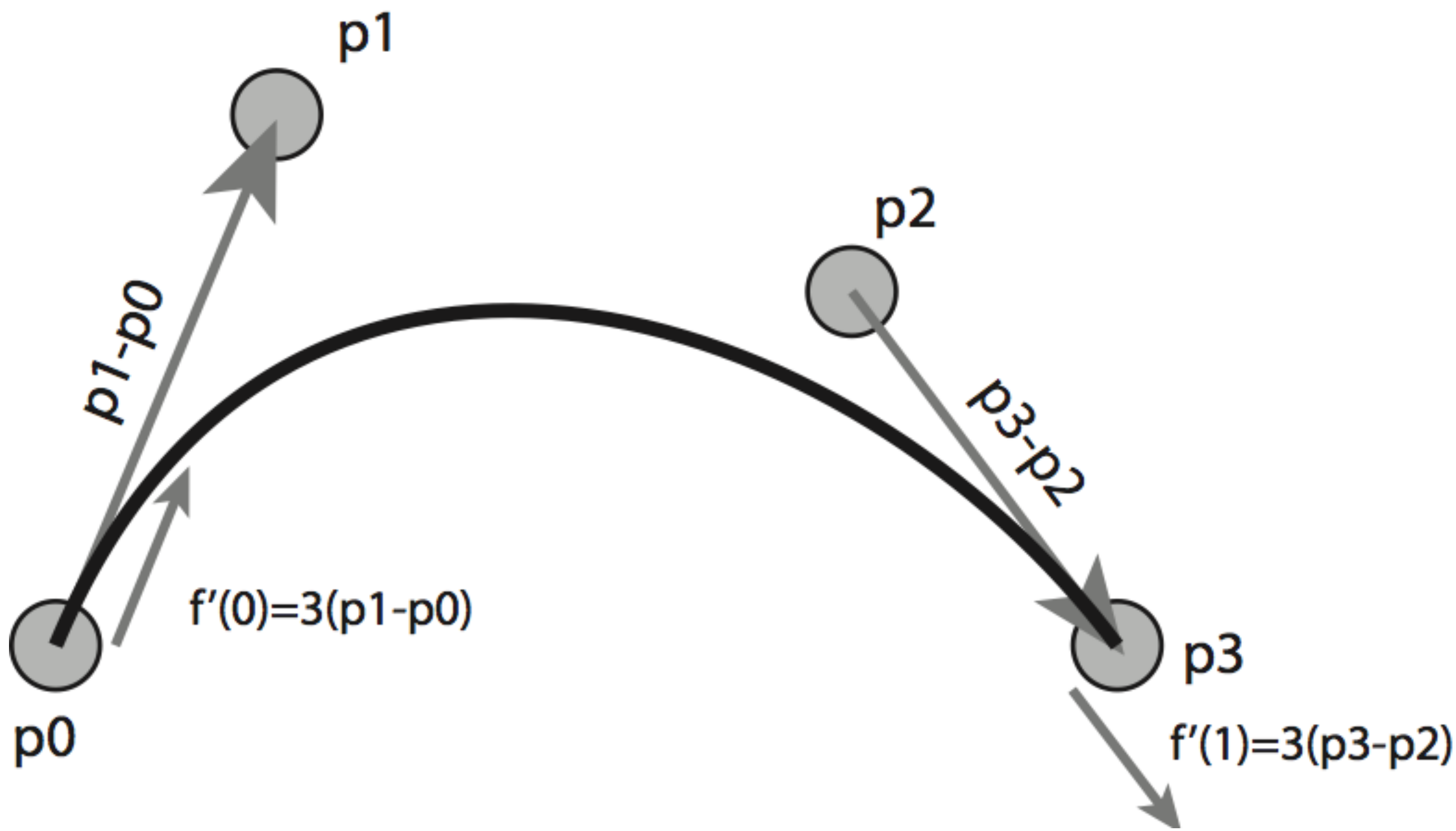
$$b_3(u) = u^3 - u^2$$

# Example: keynote curve tool

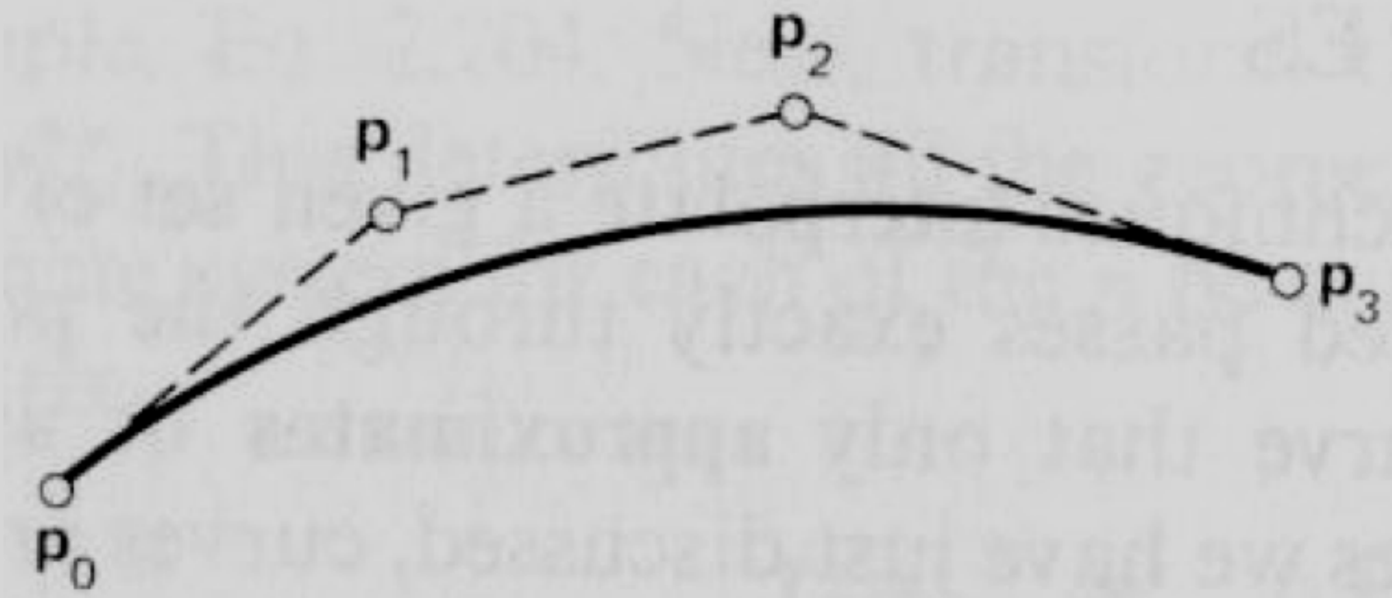


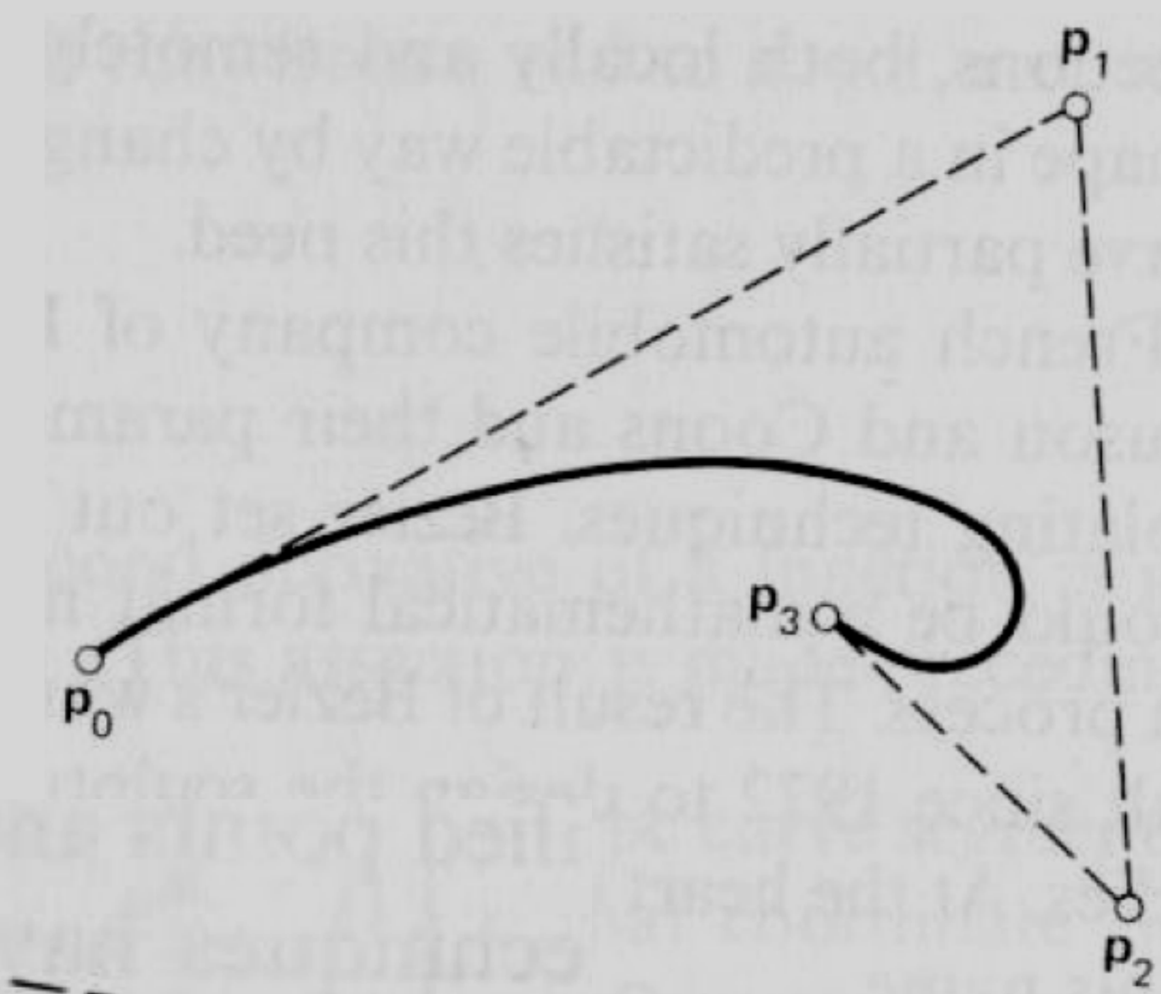
# Cubic Bezier Curves

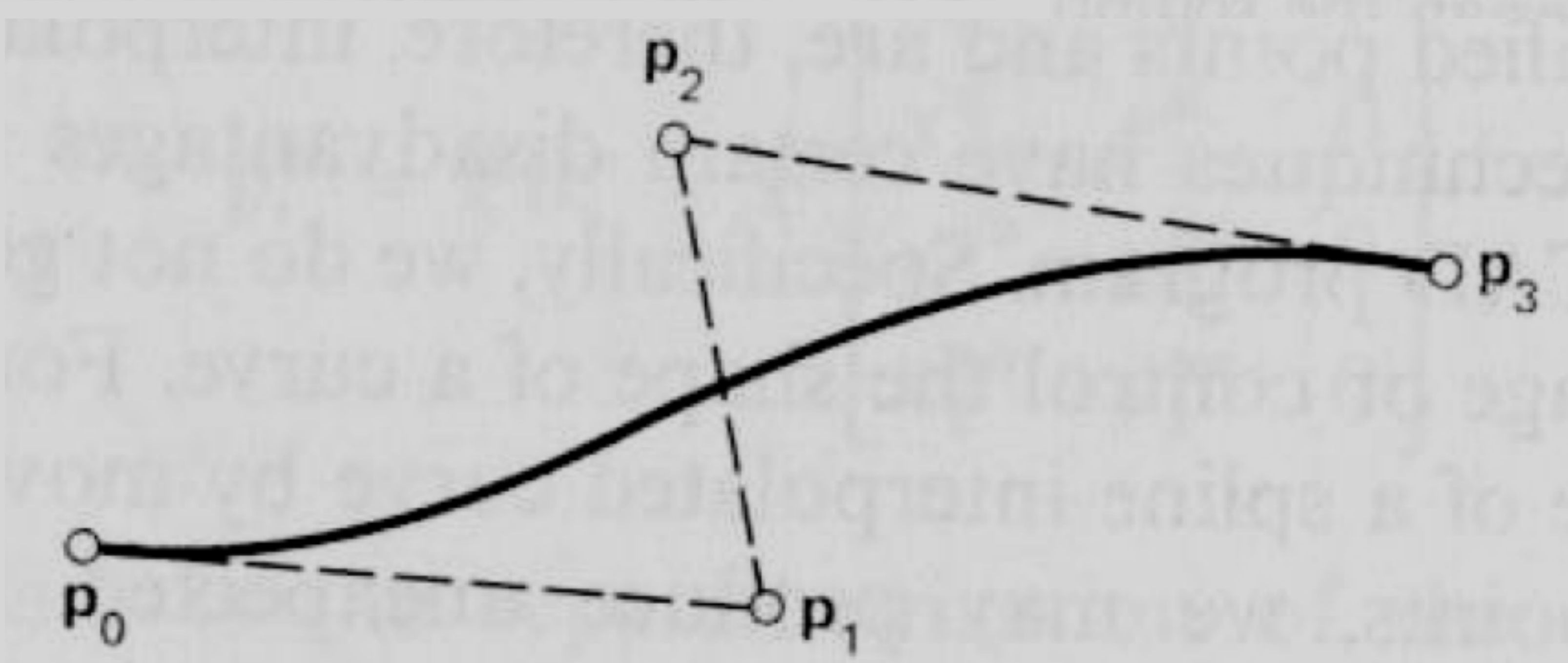
# Cubic Bezier Curves



# Cubic Bezier Curve Examples



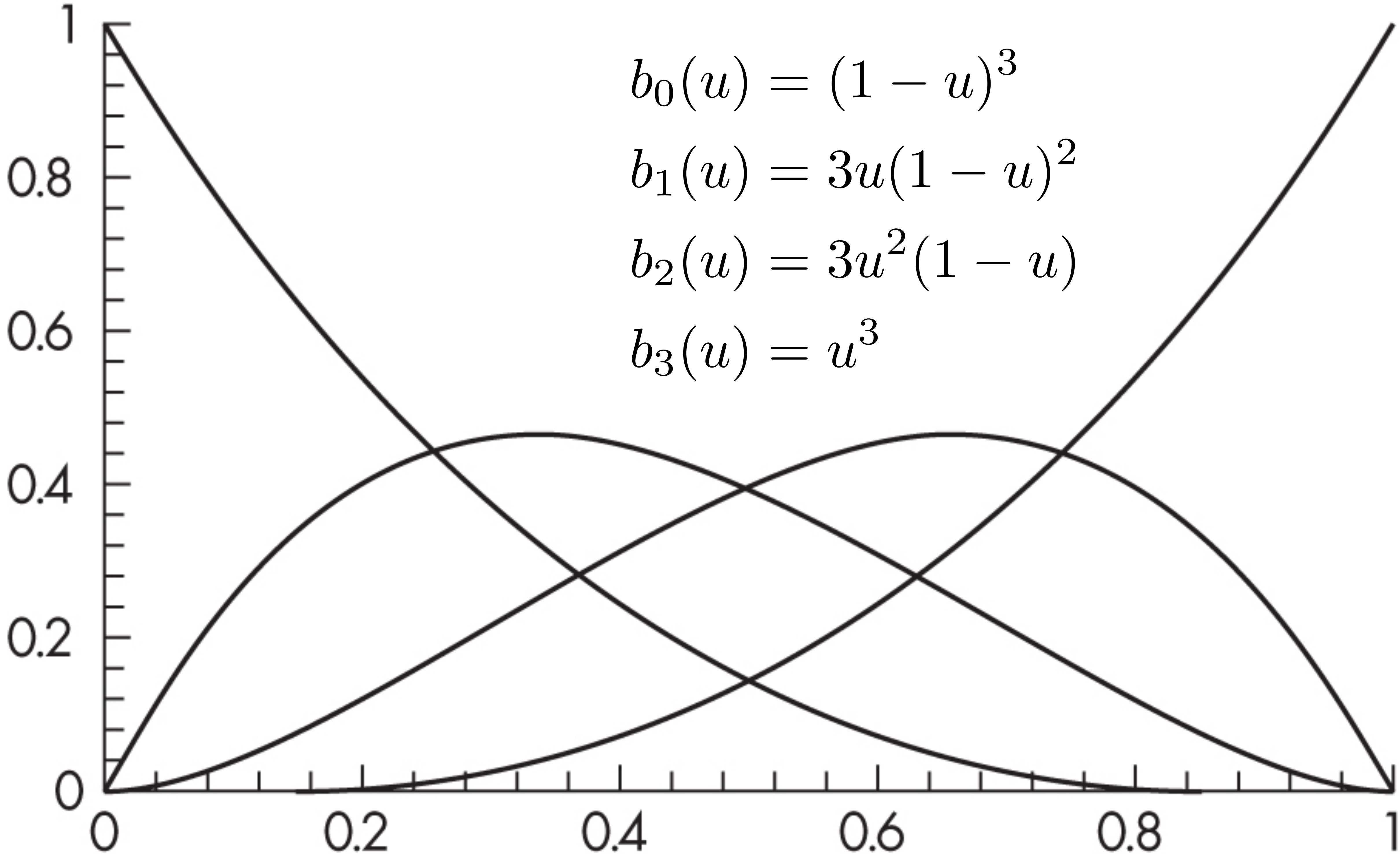



# Cubic Bezier blending functions

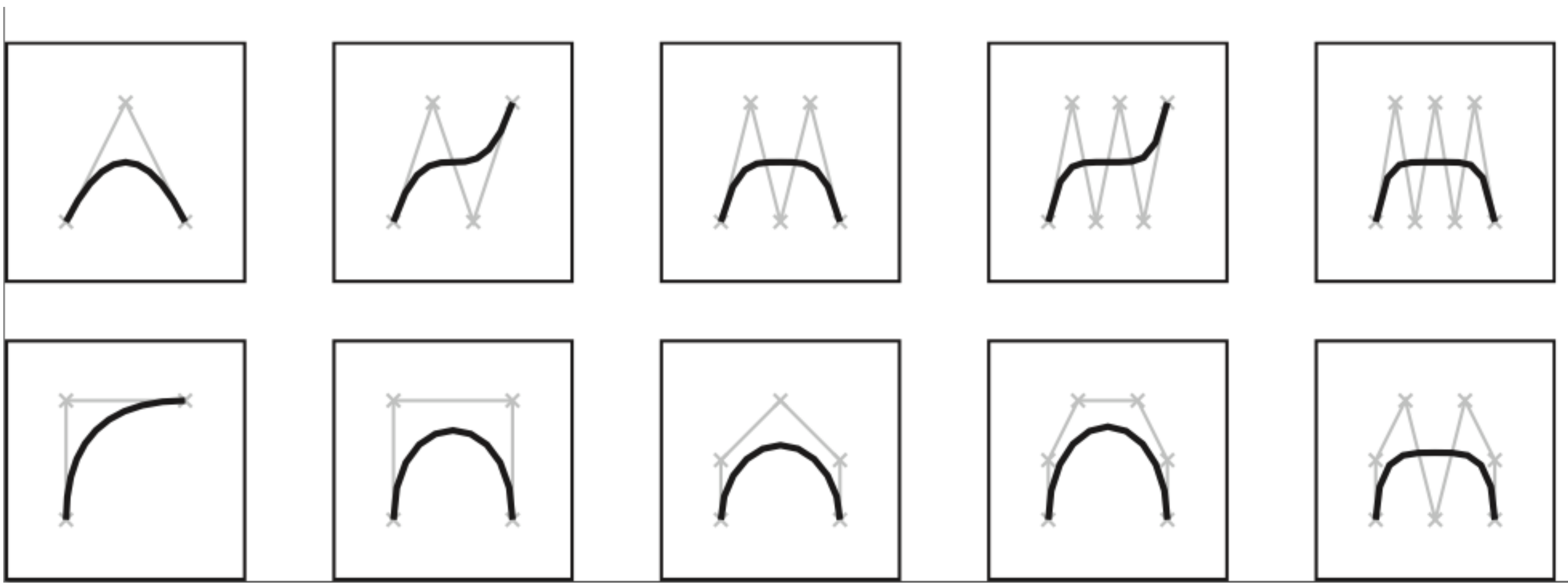
<whiteboard>

# Cubic Bezier blending functions




$$b_0(u) = (1-u)^3$$
$$b_1(u) = 3u(1-u)^2$$
$$b_2(u) = 3u^2(1-u)$$
$$b_3(u) = u^3$$

# Bezier Curves Degrees 2-6



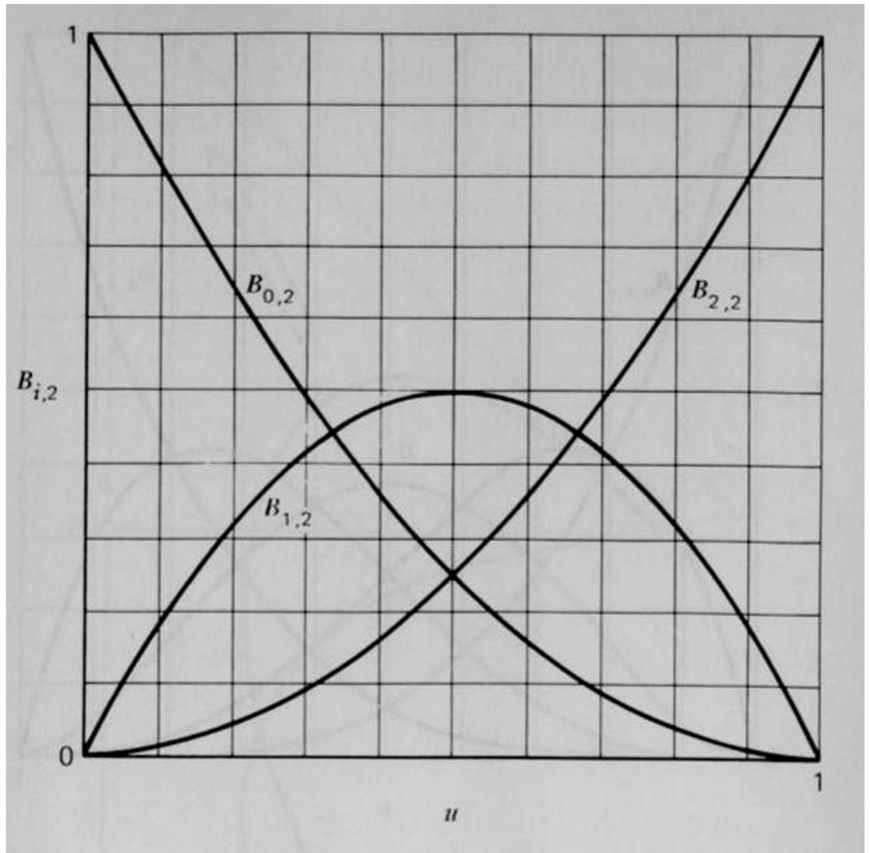
# Bernstein Polynomials

- The blending functions are a special case of the Bernstein polynomials

$$b_{\mathrm{kd}}(u) = \frac{d!}{k!(d-k)!} u^k (1-u)^{d-k}$$

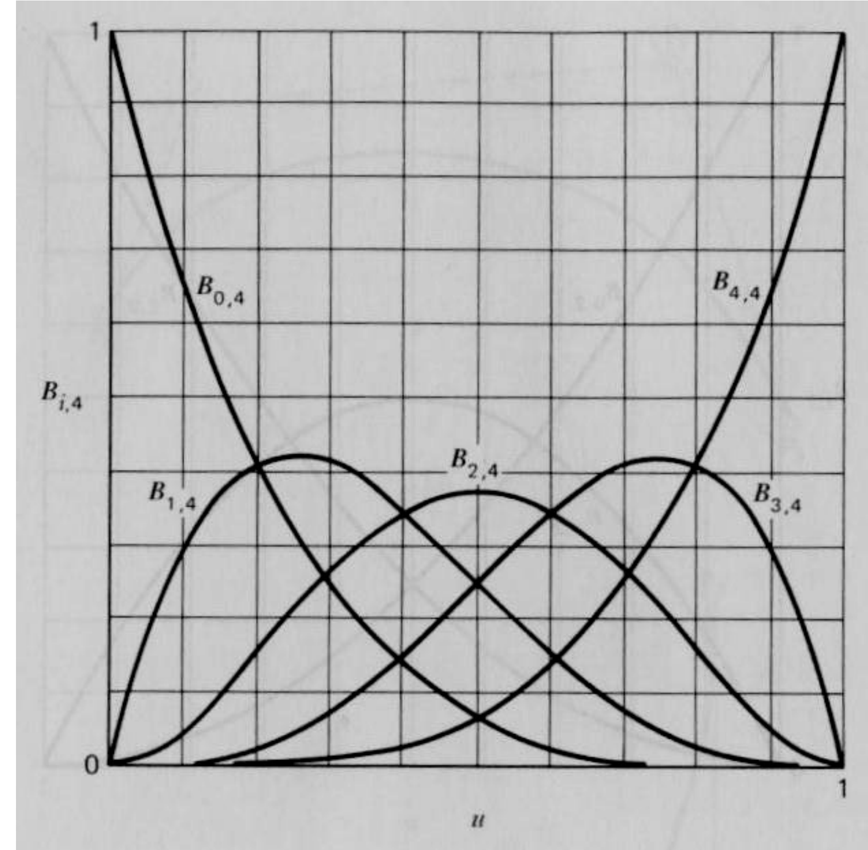
- These polynomials give the blending polynomials for any degree Bezier form
  - All roots at 0 and 1
  - For any degree they all sum to 1
  - They are all between 0 and 1 inside (0,1)

d = 2



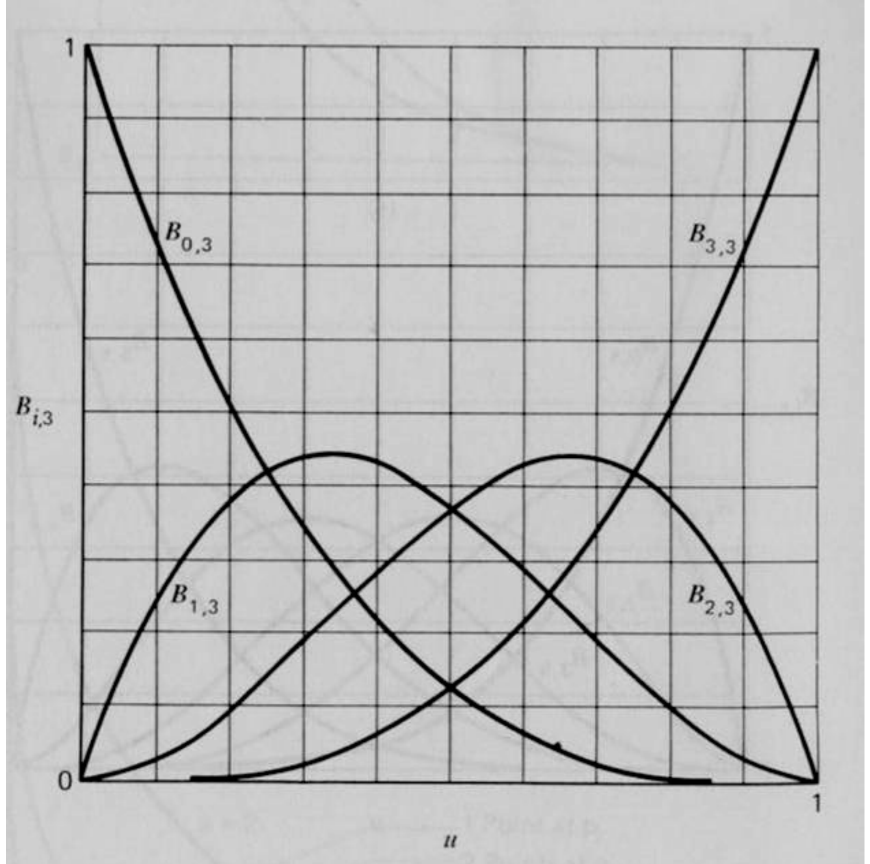

(a)

d = 4



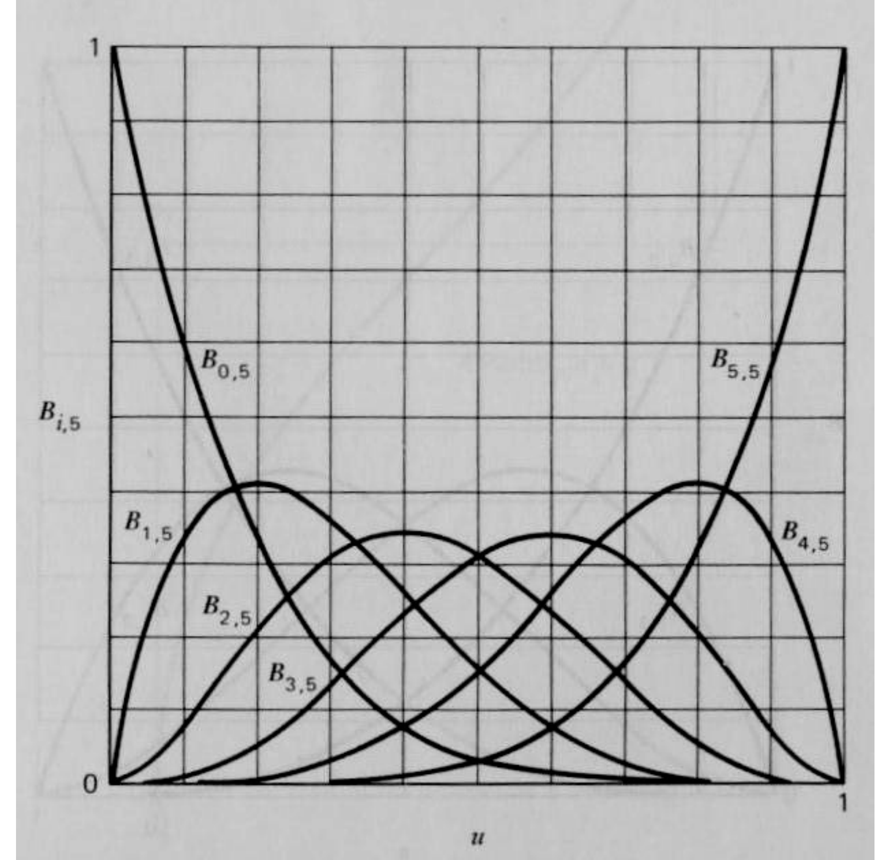

(c)

d = 3




(b)

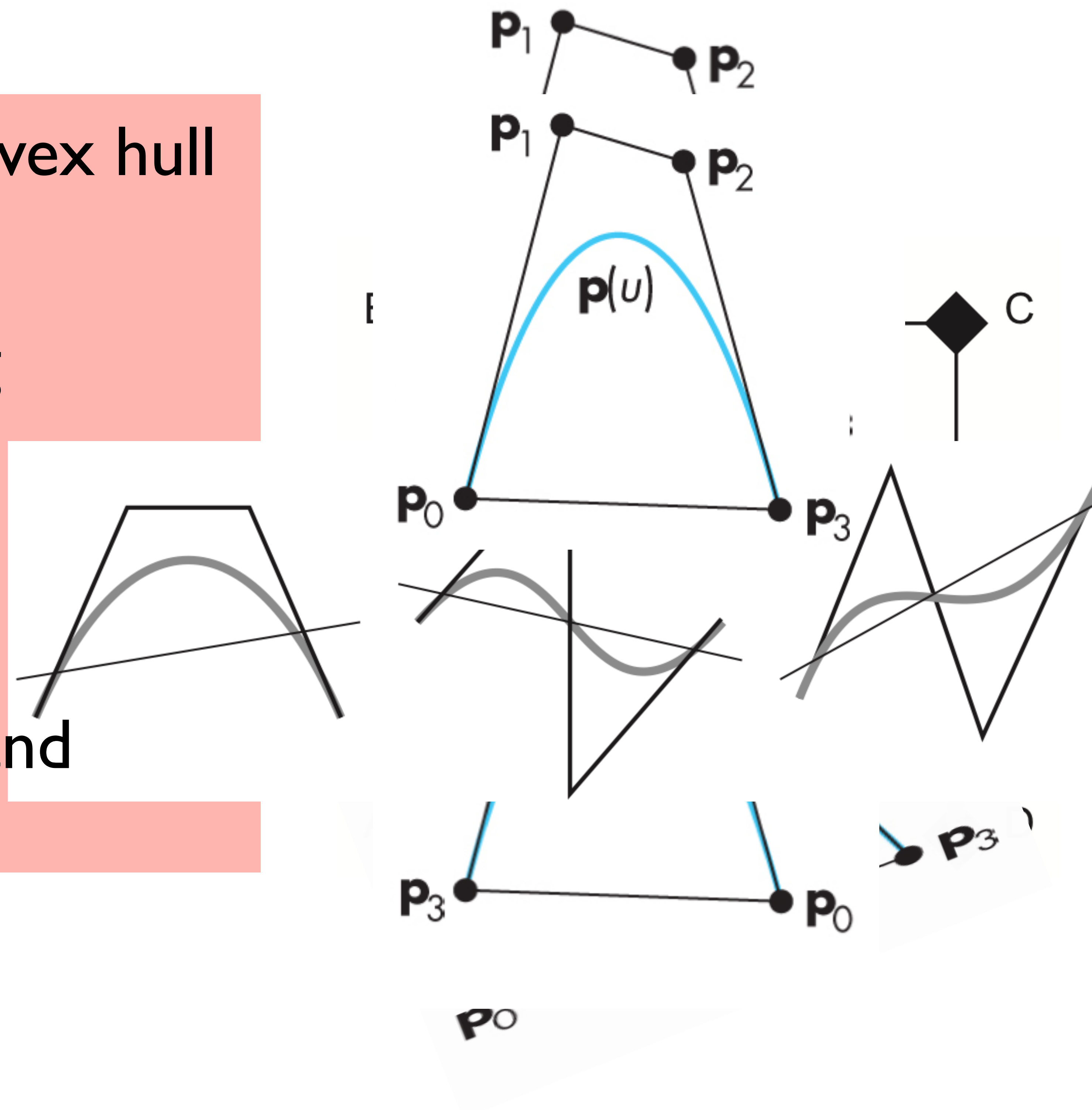
d = 5



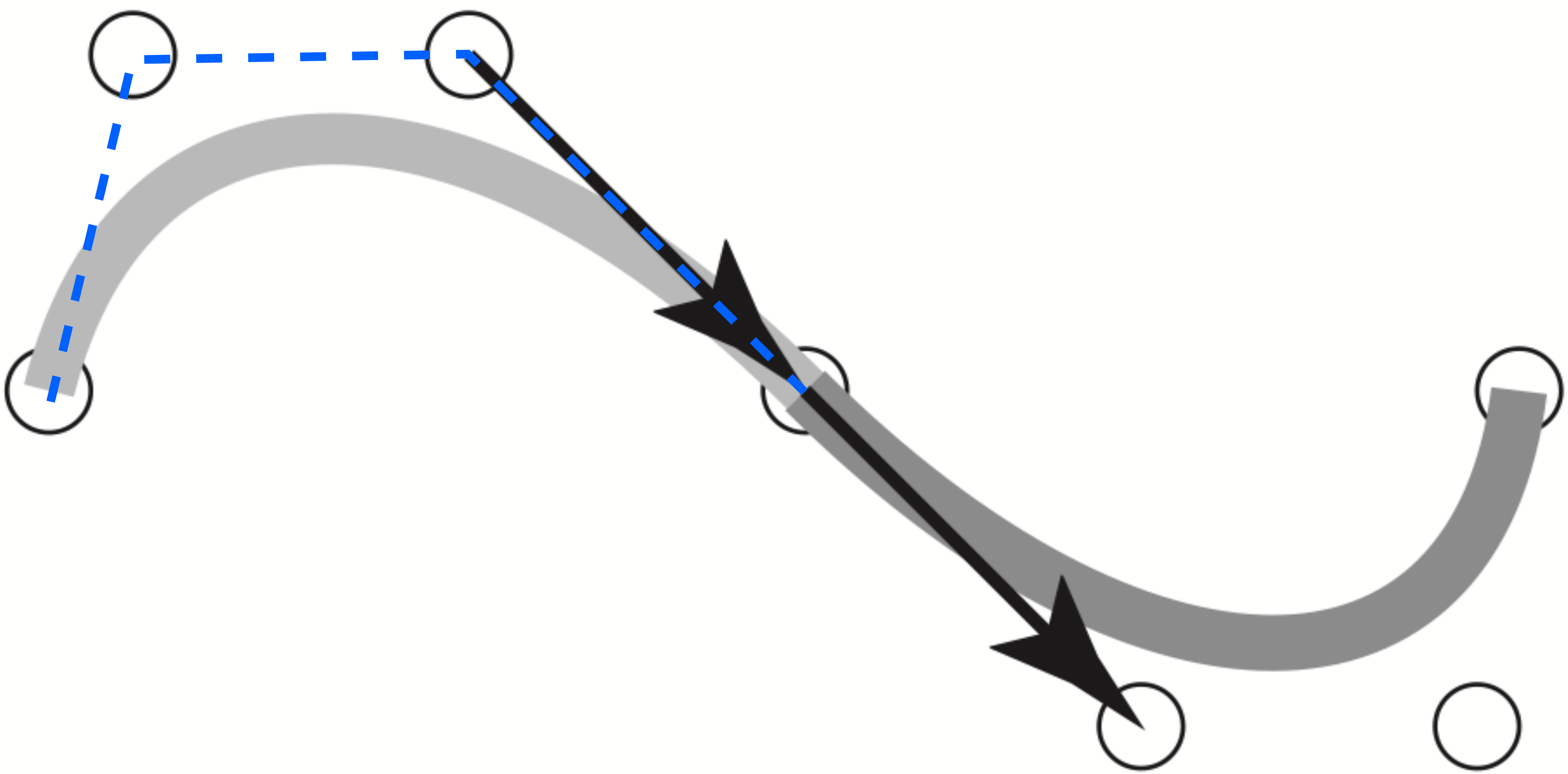

(d)

# Bezier Curve Properties

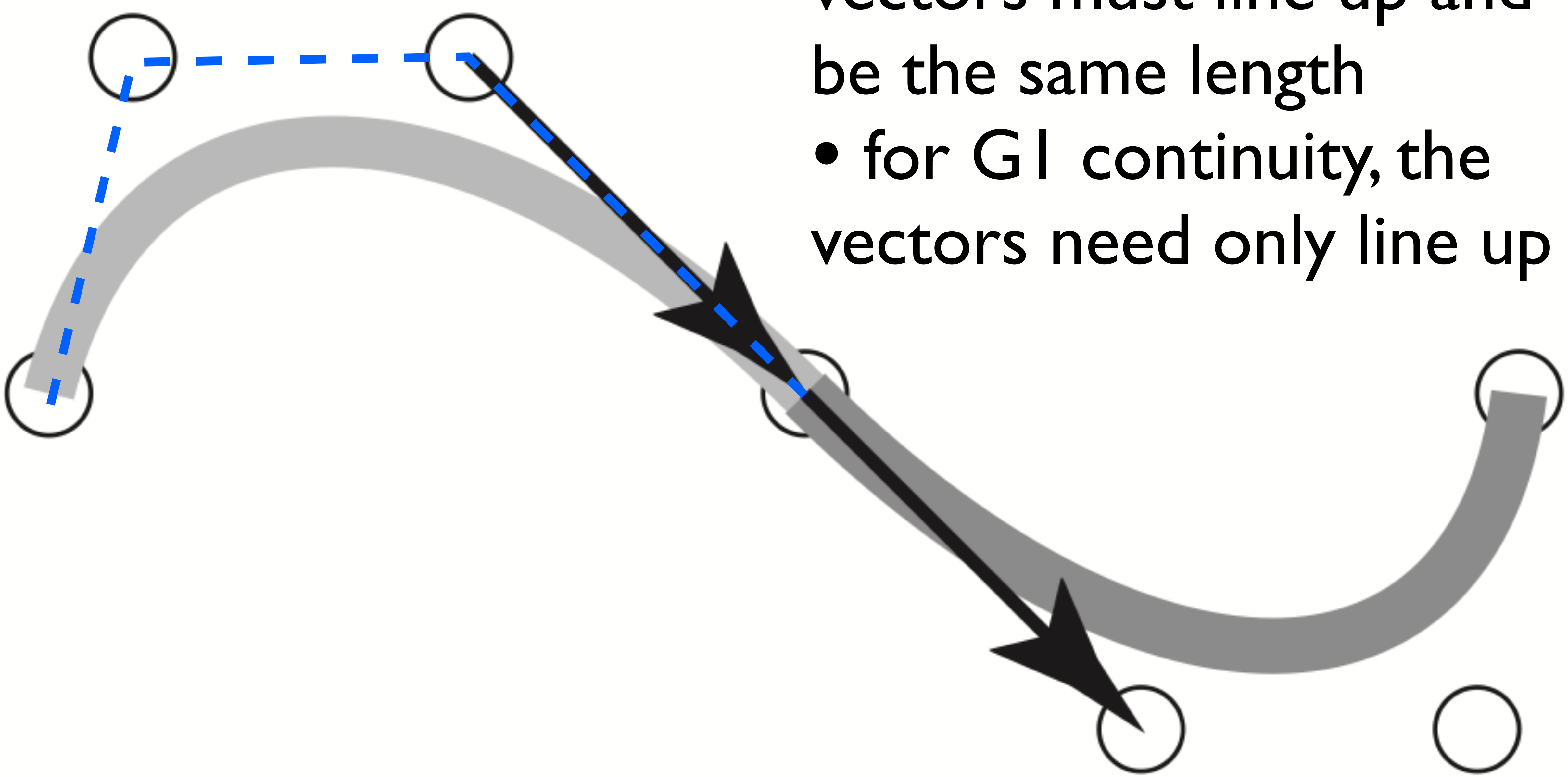
- curve lies in the convex hull of the data
- variation diminishing
- symmetry
- affine invariant
- efficient evaluation and subdivision



# Joining Cubic Bezier Curves

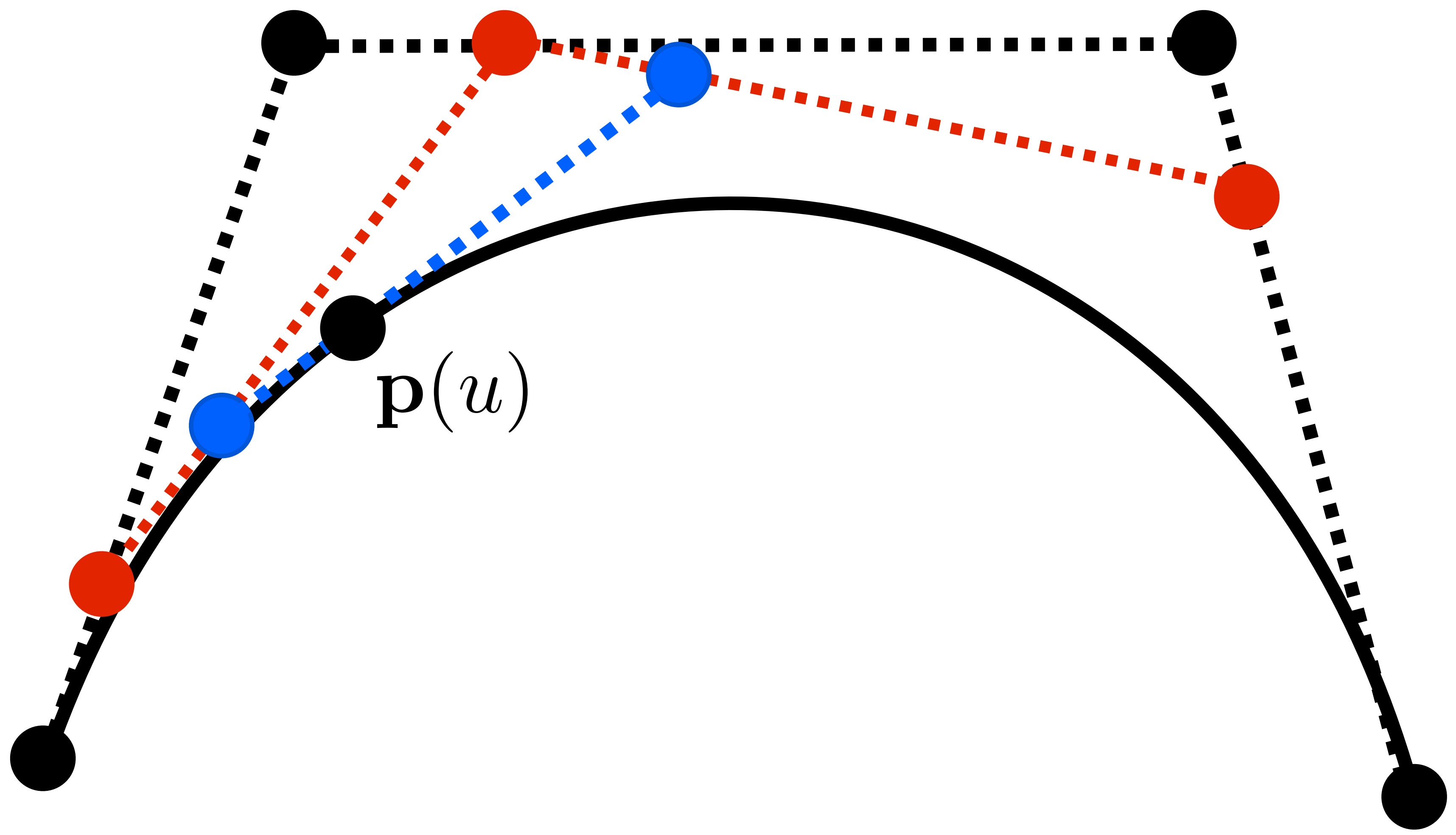


# Joining Cubic Bezier Curves

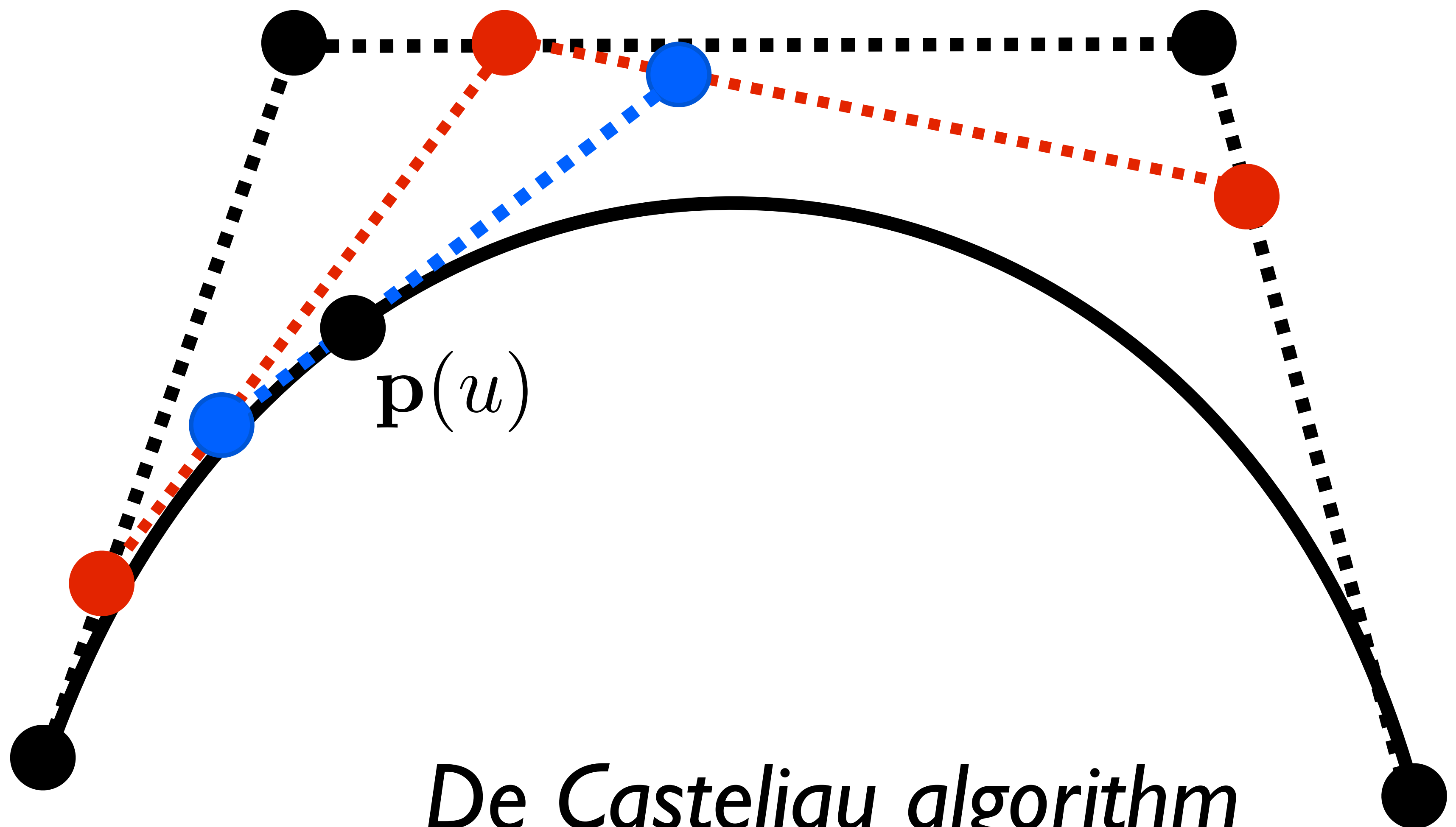


- for C1 continuity, the vectors must line up and be the same length
- for G1 continuity, the vectors need only line up

# Evaluating p(u) geometrically

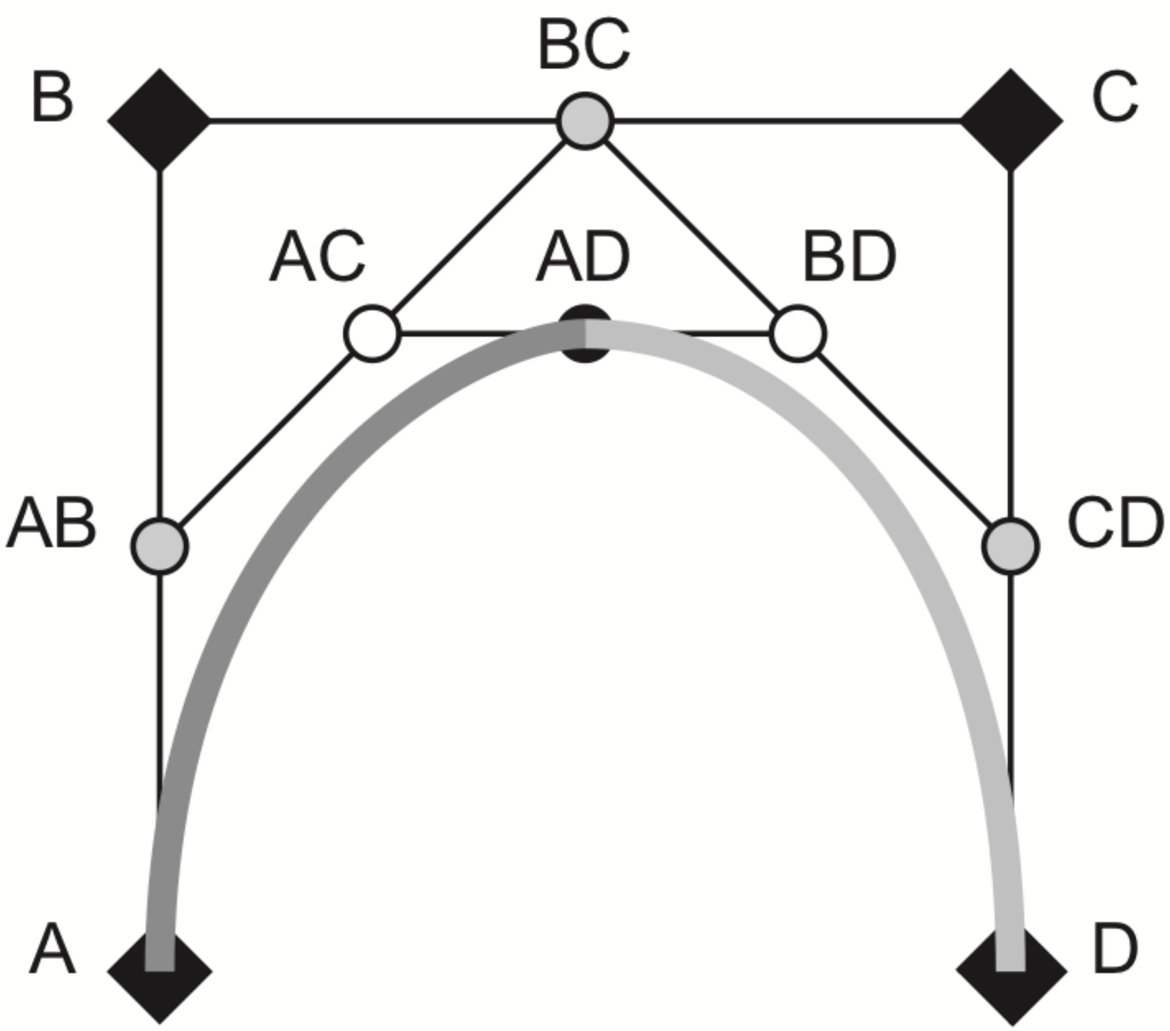


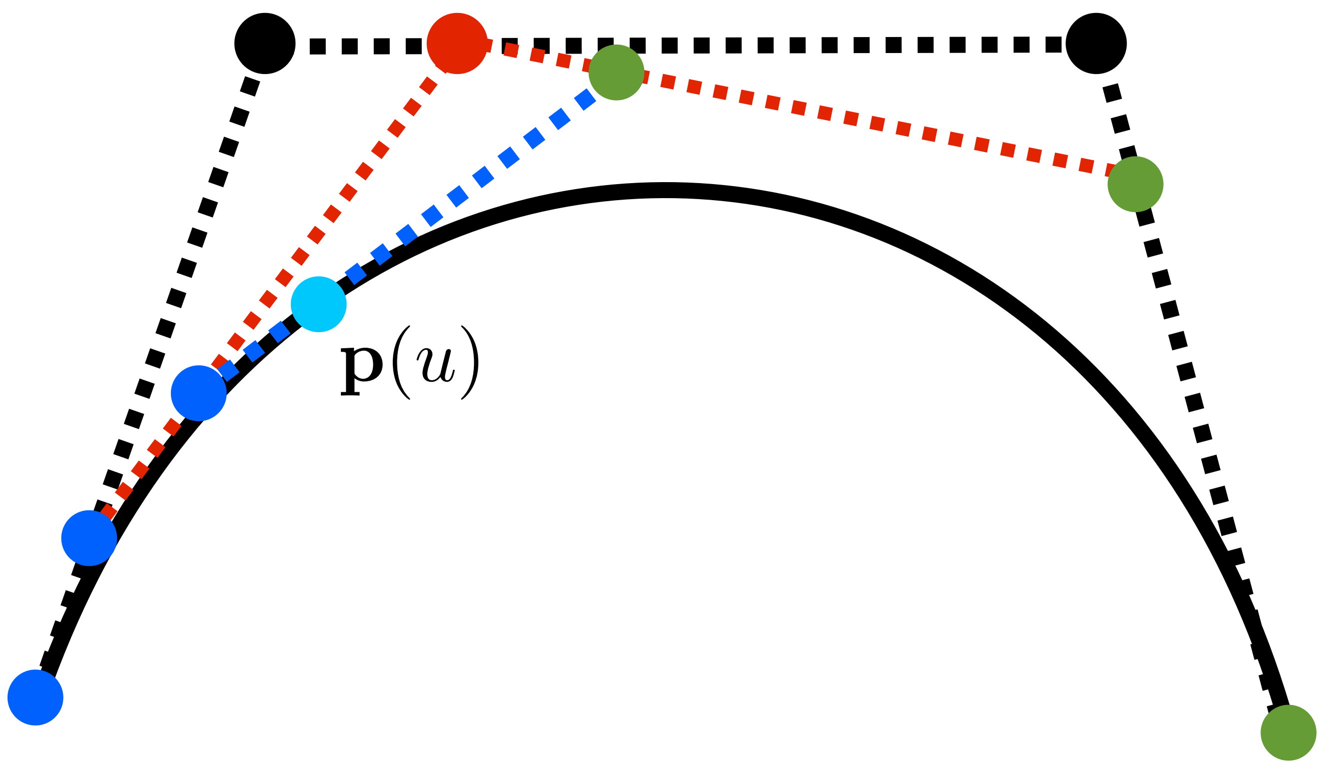
# Evaluating p(u) geometrically



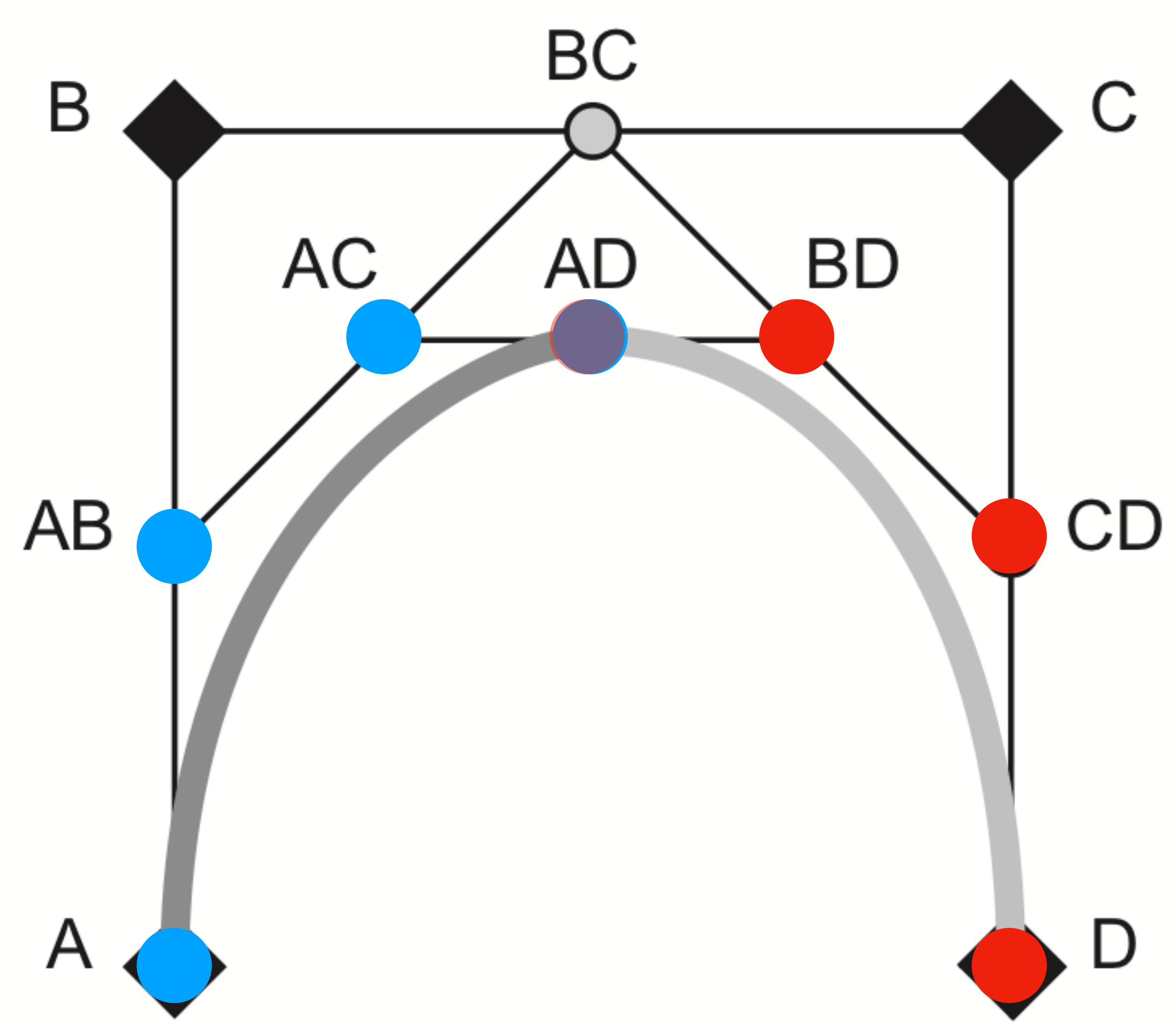

*De Casteljau algorithm*

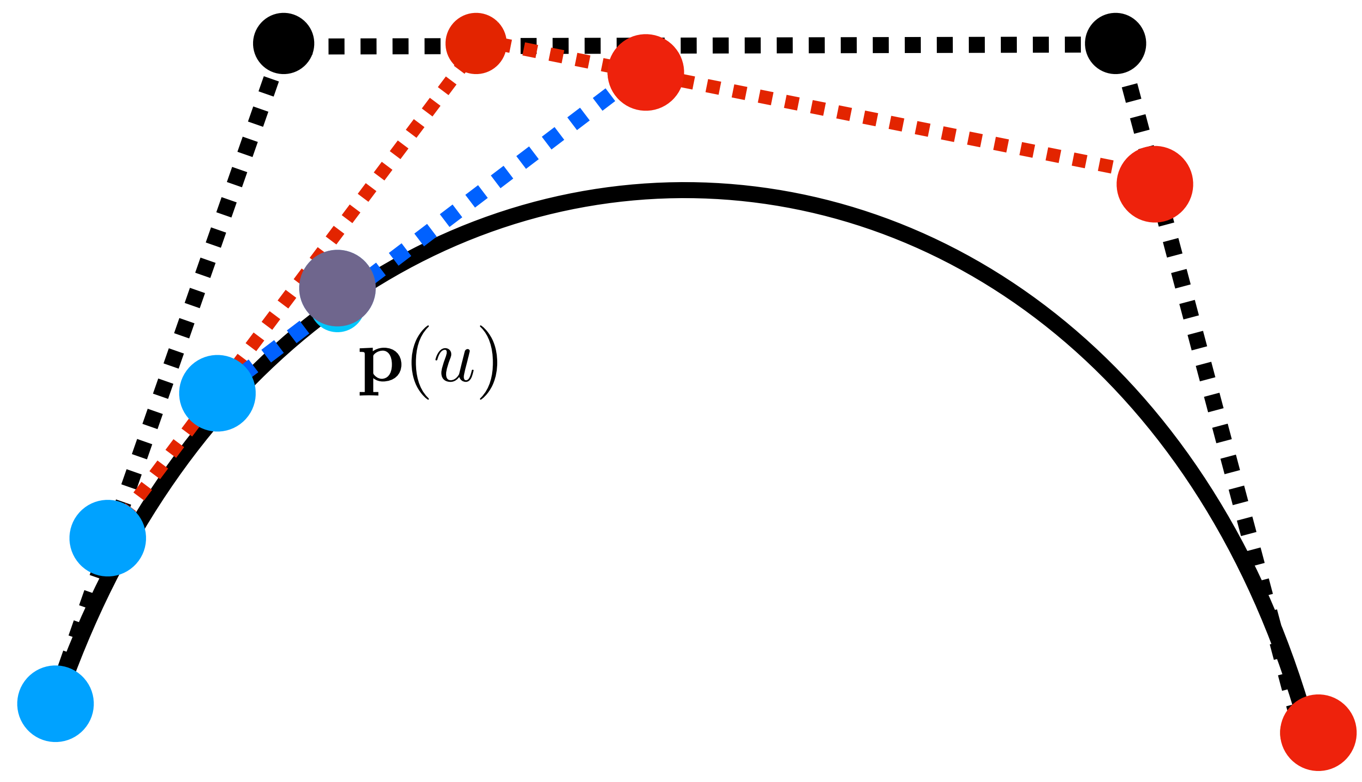
# Bezier subdivision



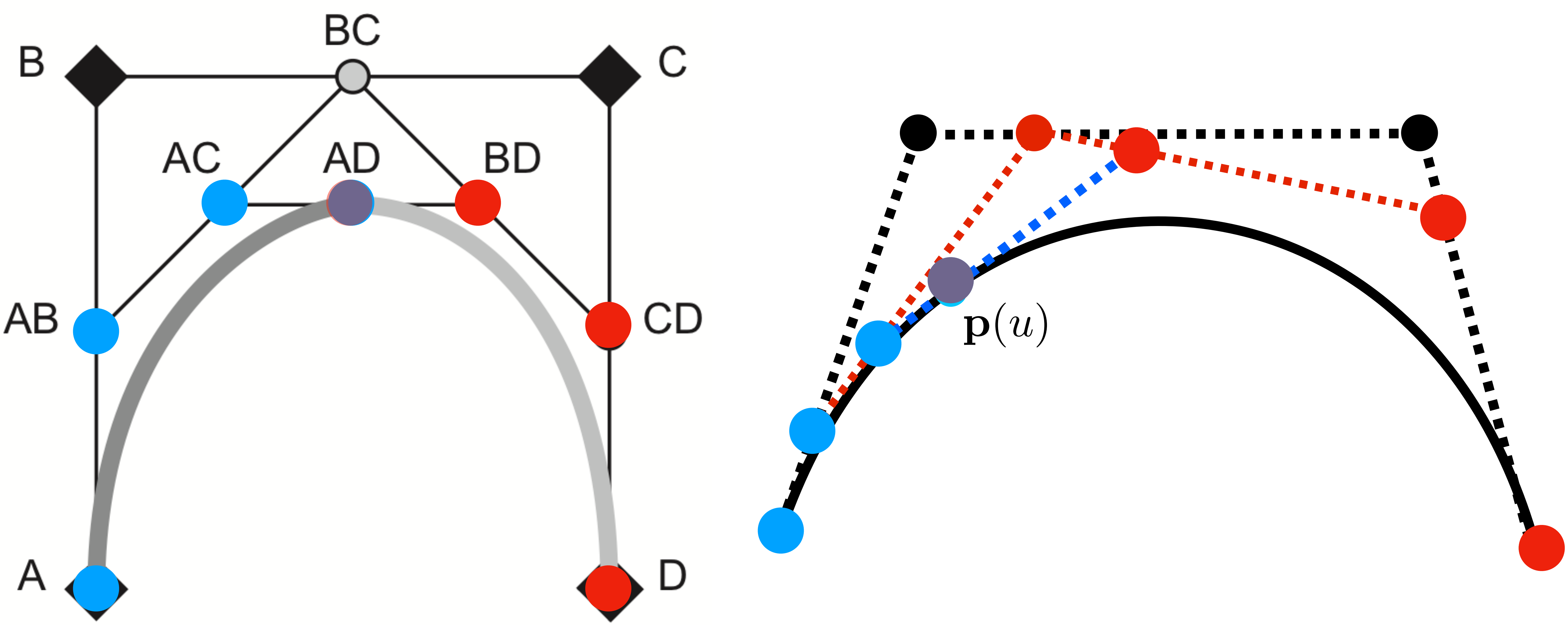



# Bezier subdivision



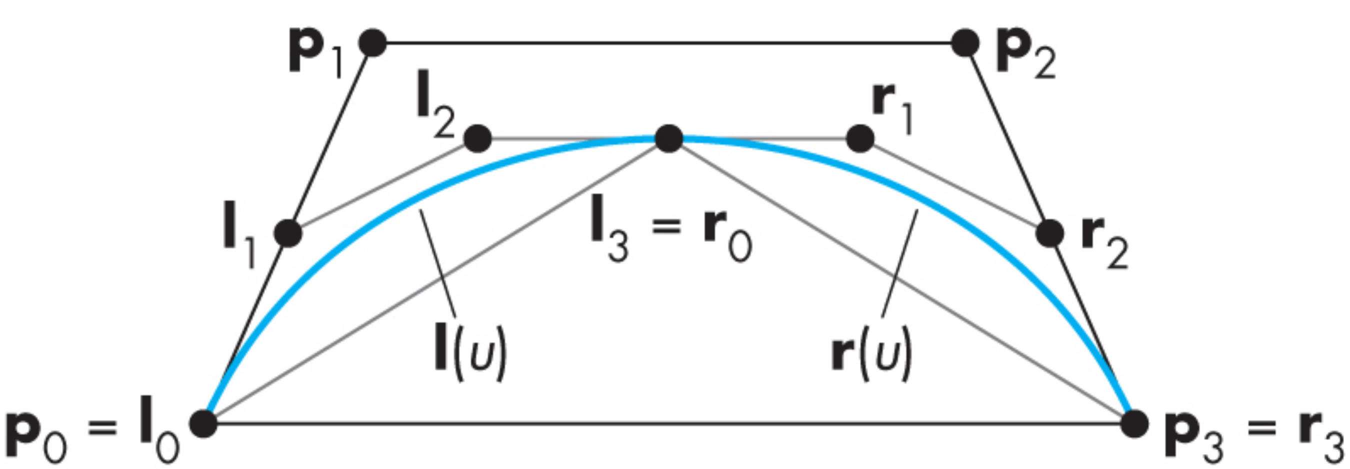
# Bezier subdivision

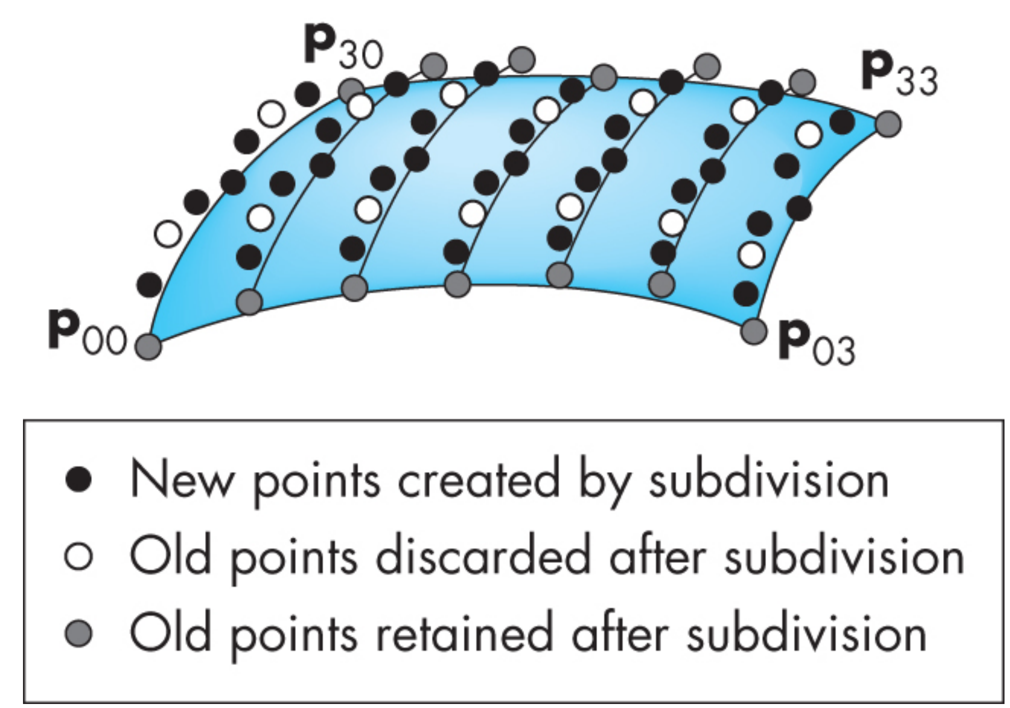



divid and conquer approach can be
used for efficient rendering

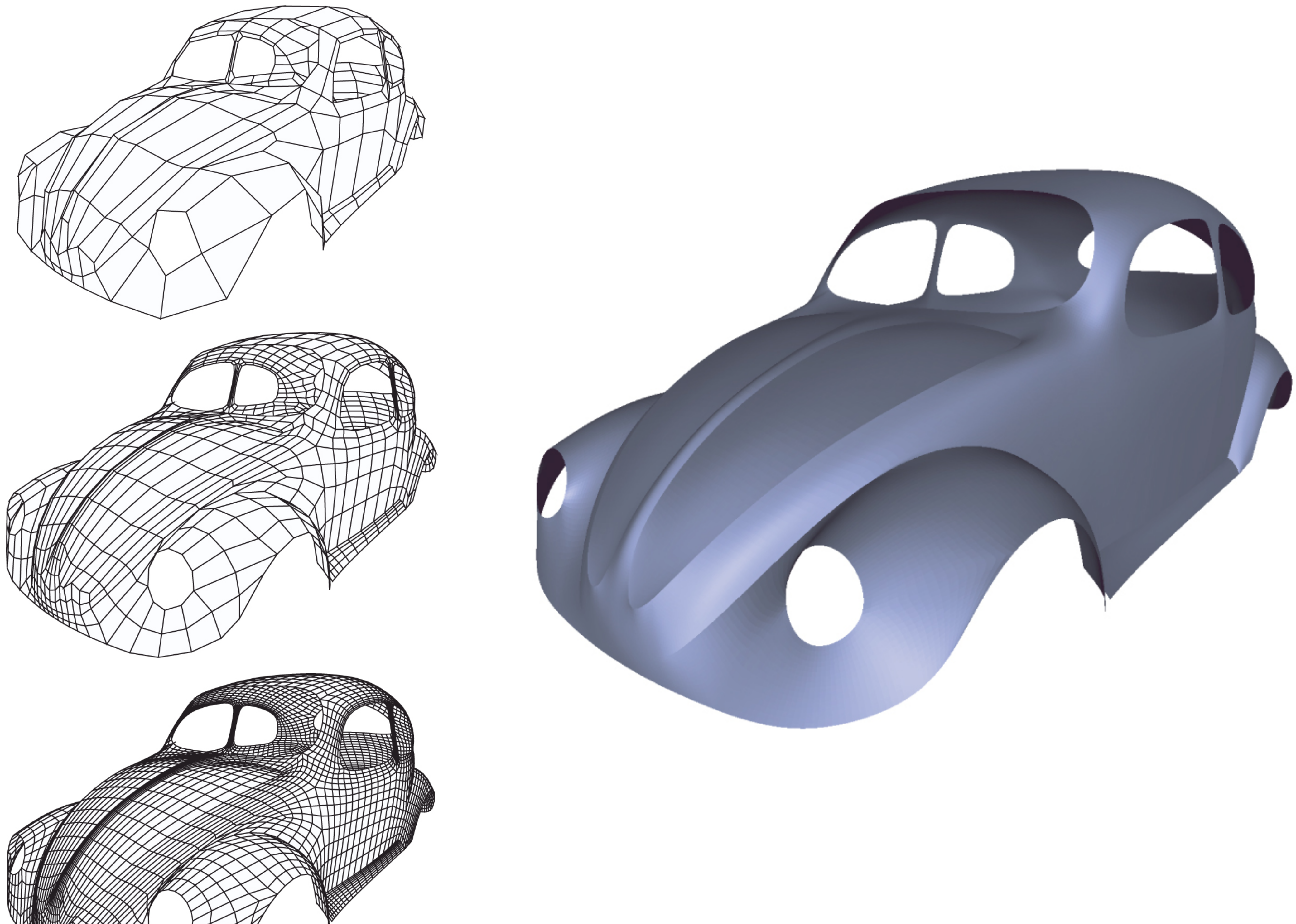
# Recursive Subdivision

- work with convex hull, does not require evaluating the polynomial
- Bezier curves most convenient -- other curves can be transformed to Bezier
- same approach for surfaces



# Recursive Subdivision for Rendering



# Cubic B-Splines

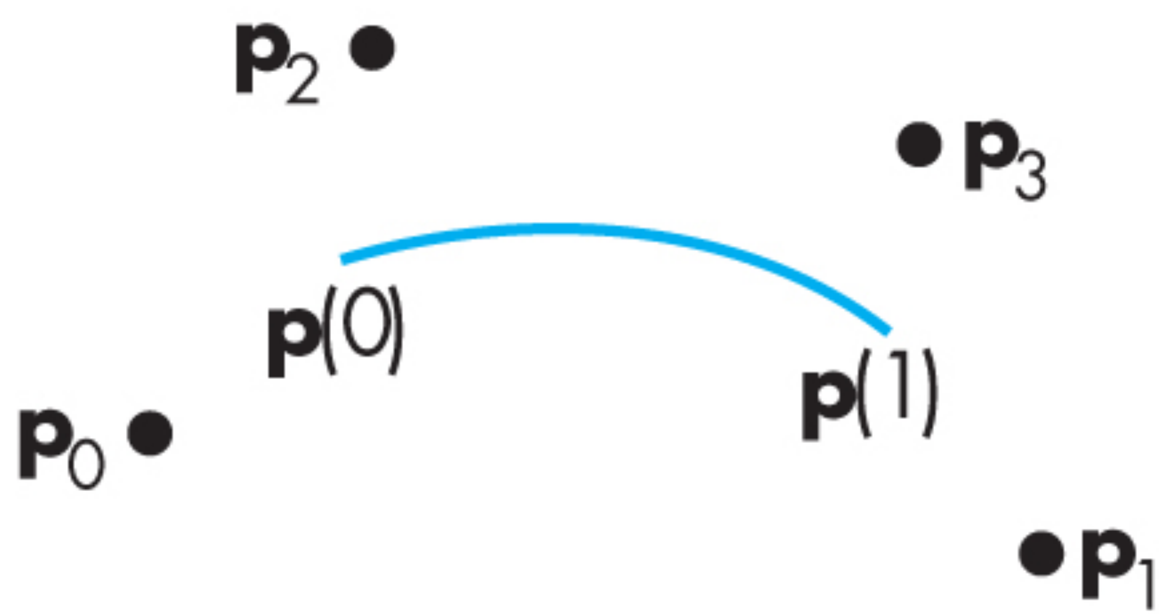
# B-spline properties

- polynomials of degree d with (d-1) continuity
- preferred method for very smooth curves (C2 or higher)

# B-spline properties

- C(d-1) continuity
- local control - any point on curve depends on d+1 control points
- bounded by convex hull
- variation diminishing property

# Cubic B-Splines



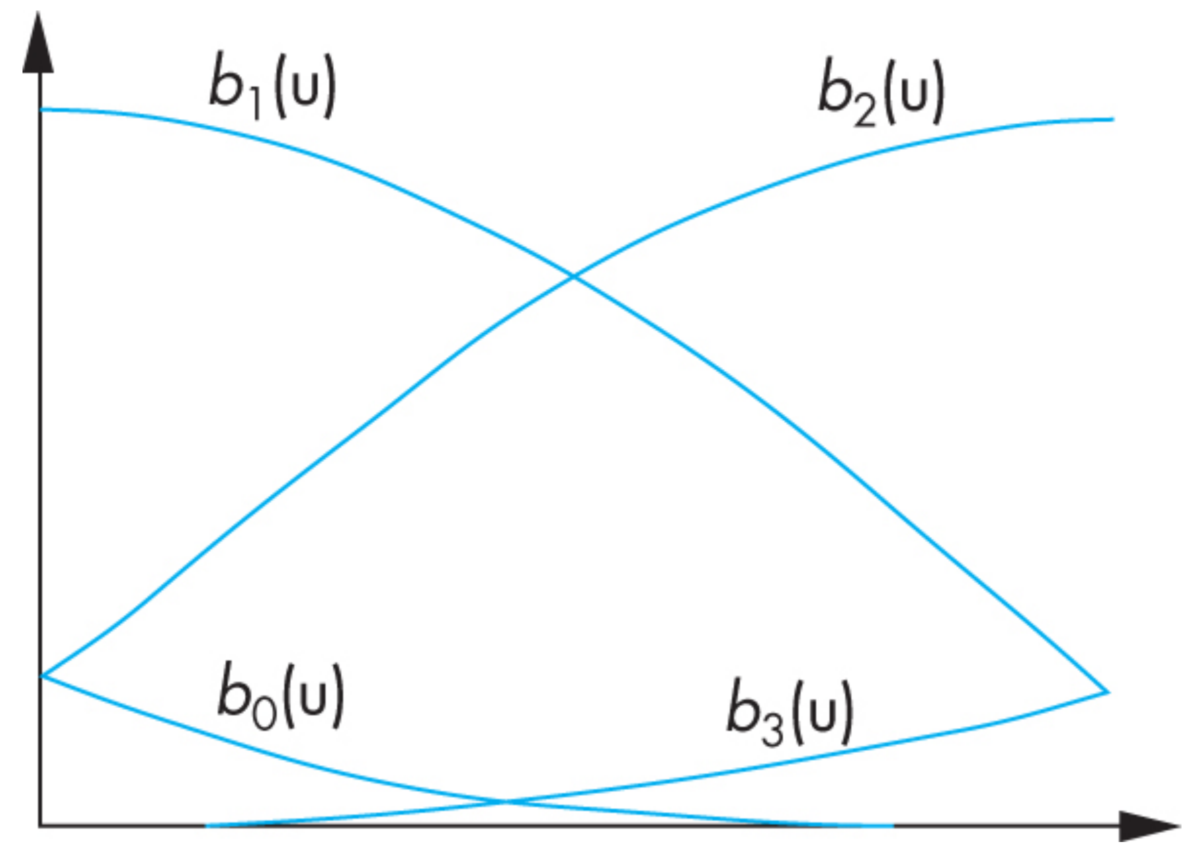
# Spline blending functions

$$b_0(u) = \frac{1}{6}(1-u)^3$$

$$b_1(u) = \frac{1}{6}(4 - 6u^2 + 3u^3)$$

$$b_2(u) = \frac{1}{6}(1 + 3u + 3u^2 - 3u^3)$$

$$b_3(u) = \frac{1}{6}u^3$$

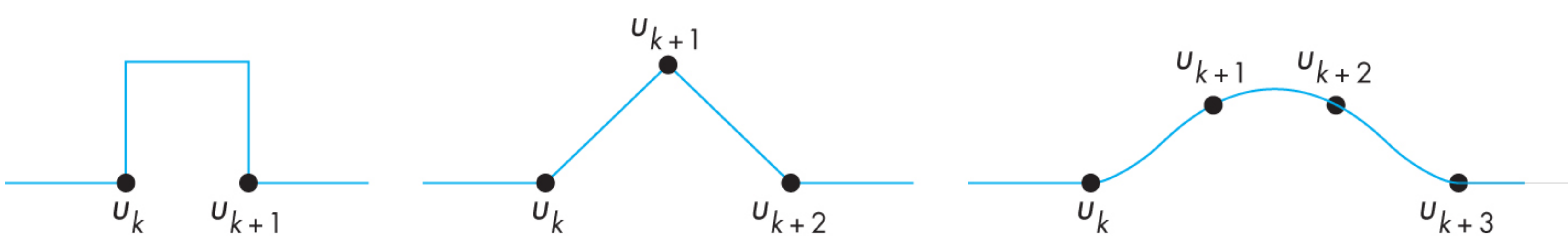


# General Splines

- Defined recursively by *Cox-de Boor recursion formula*

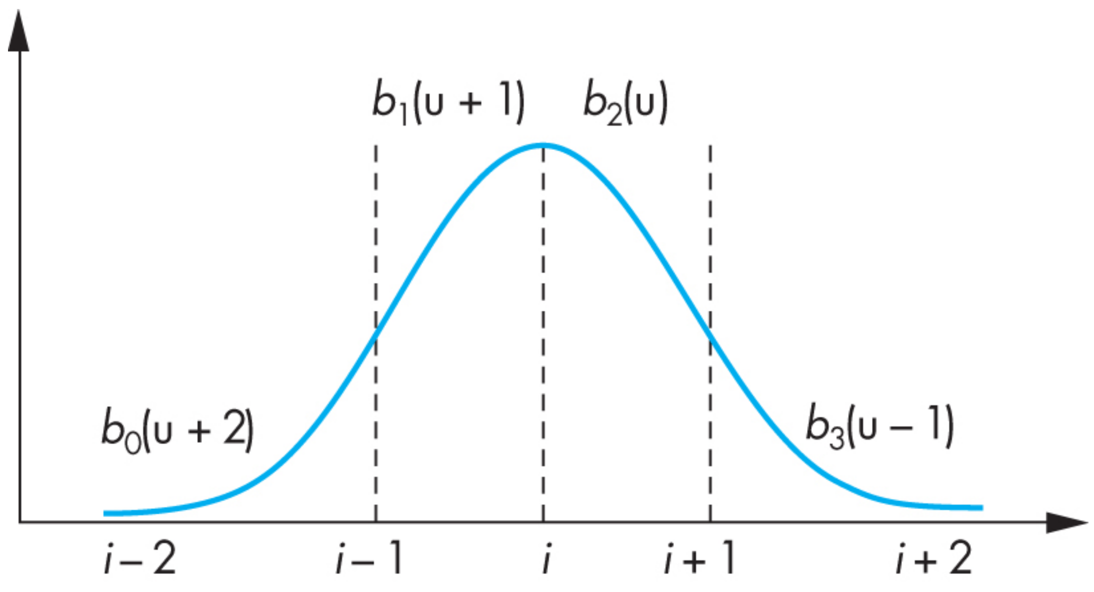
$$b_{j,0}(t) = \begin{cases} 1 & \text{if} \quad t_j \leq t \\ 0 & \text{otherwise} \end{cases}$$

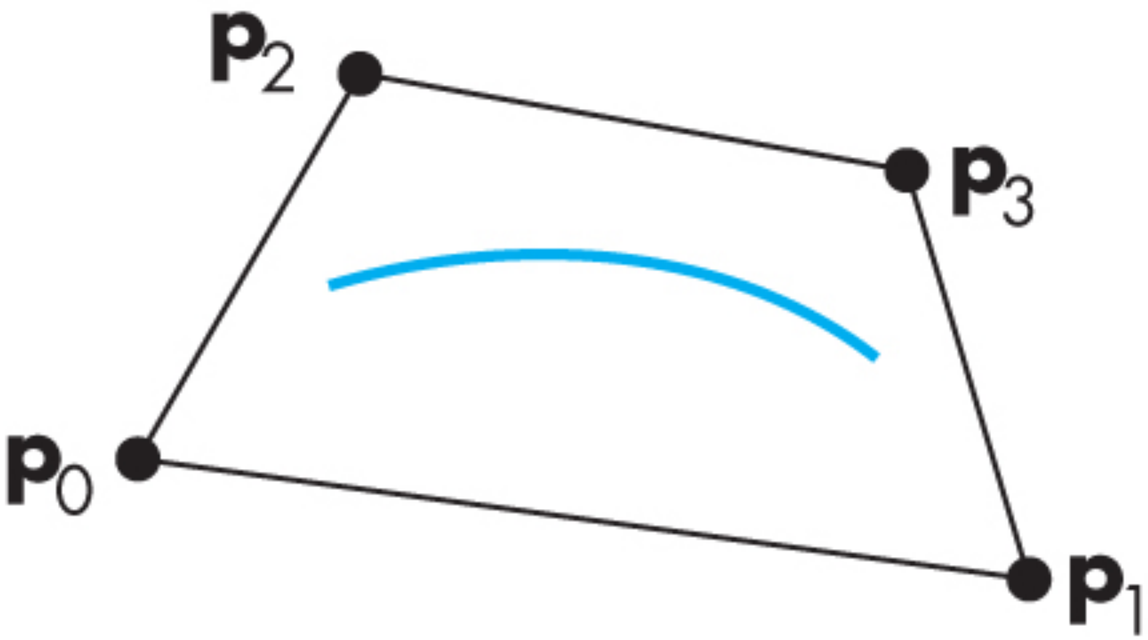
$$b_{j,n}(t) := \frac{t - t_j}{t_{j+n} - t_j} b_{j,n-1}(t) + \frac{t_{j+n+1} - t}{t_{j+n+1} - t_{j+1}} b_{j+1,n-1}(t)$$



# Spline properties

Basis functions



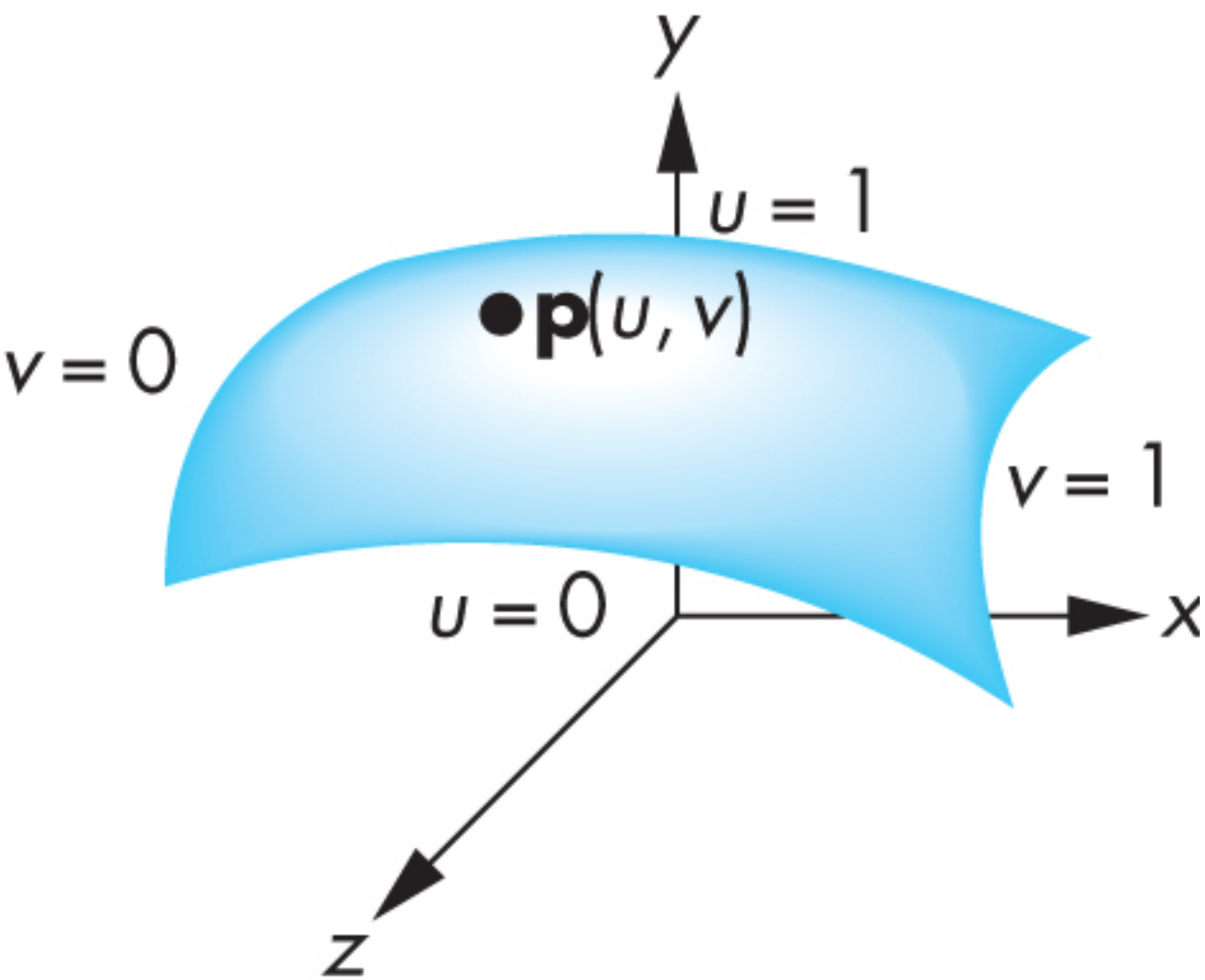




convexity

# Surfaces

# Parametric Surface

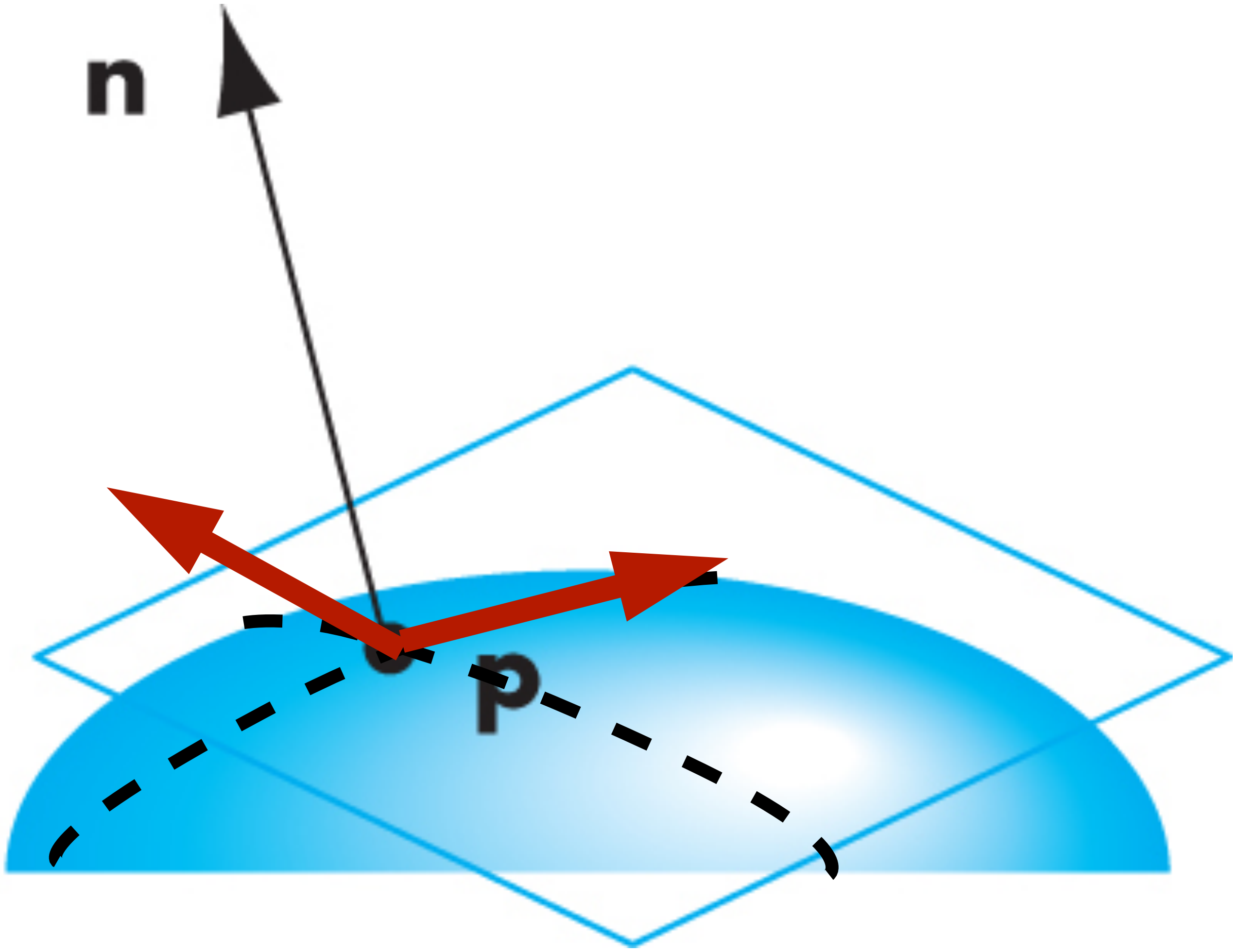
$$x = x(u, v)$$
$$y = y(u, v)$$
$$z = z(u, v)$$



# Parametric Surface - tangent plane

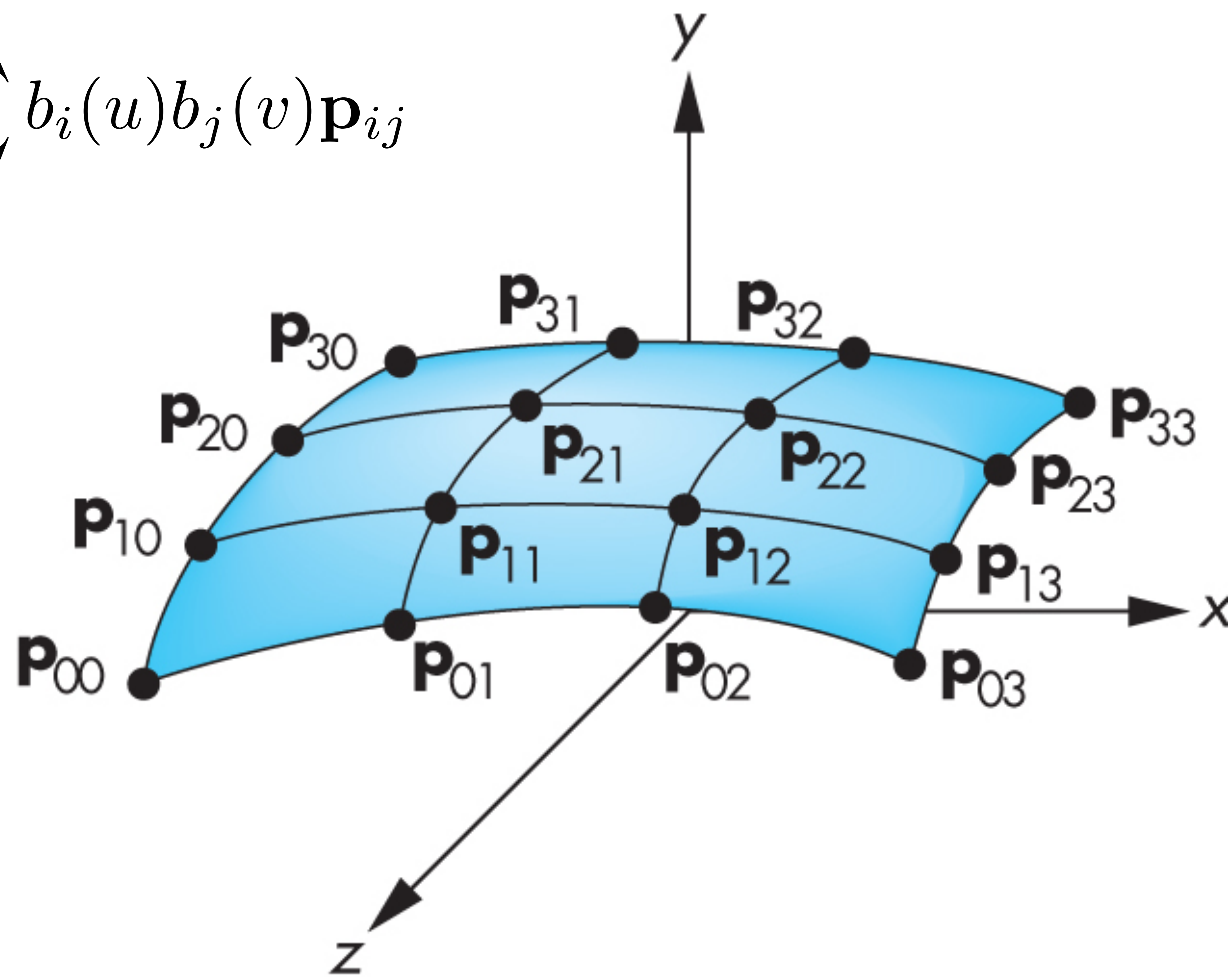
$$\mathbf{t}_u = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}$$

$$\mathbf{t}_v = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}$$



# Bicubic Surface Patch

$$\mathbf{f}(u,v) = \sum_i \sum_j b_i(u) b_j(v) \mathbf{p}_{ij}$$



# Bezier Surface Patch

$$\mathbf{f}(u, v) = \sum_i \sum_j b_i(u) b_j(v) \mathbf{p}_{ij}$$

Patch lies in convex hull