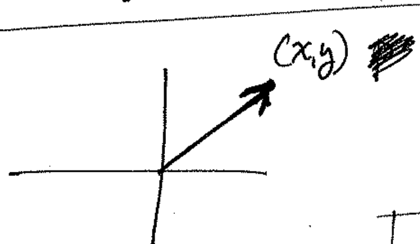
LECTURE 7

6.1.1-6.1.5
6.2.0-6.2.1
6.3
-6.5

geometric transformations:
- rotation
- translation
- scaling
- projection
- shear

} can be accomplished with <u>transformation</u> <u>matrices</u>

6.1 2D <u>linear</u> transformation



a point represented as an offset from the origin.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

2D linear transformation

(*) Linear
$A(\underline{u}+\underline{v}) = A\underline{u}+A\underline{v}$
$A(\alpha\underline{u}) = \alpha A\underline{u}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

A maps one 2D vector to another.

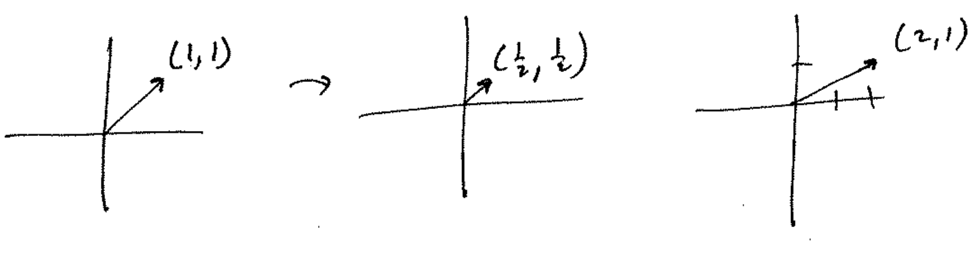
Choice of A <u>entries</u> yield diff. transformations.

6.1.1 Scaling

$$\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \end{pmatrix}$$

examples of uniform & non uniform

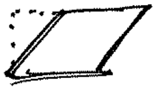
examples



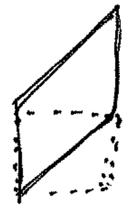
$\boxed{6.1.2}$ $\boxed{\text{Shearing.}}$

"deck of cards"

horizontal shear $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$



vertical shear

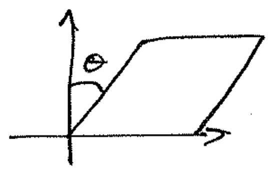


$\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$

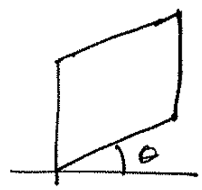
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ 45°

Generally,

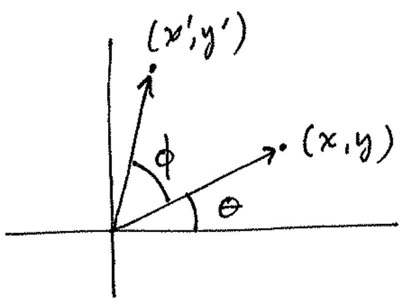
$\begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$



$\begin{pmatrix} 1 & 0 \\ \tan\theta & 1 \end{pmatrix}$



$\boxed{6.1.3}$ $\boxed{\text{Rotation}}$.



$(x, y) = (\cos\theta, \sin\theta)$

$(x', y')^T = \begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{pmatrix} = \begin{matrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \cos\theta\sin\phi + \cos\phi\sin\phi \end{matrix}$

$= \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix} =$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}}_{\text{rotate}(\phi)} \begin{pmatrix} x \\ y \end{pmatrix}$

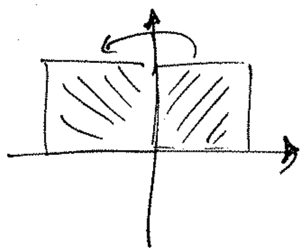
rotates counterclockwise

$R(\phi)$

to rotate clockwise by ϕ, rotate c.c.w. by $-\phi$

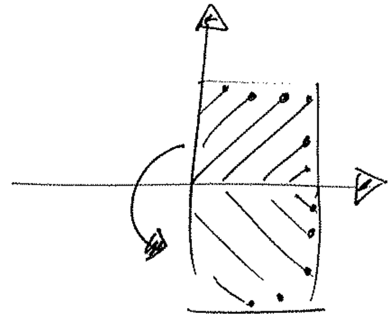
(*) Rotation matrix is an orthogonal matrix. $R(\phi)^T R(\phi) = R(\phi)R(\phi)^T = I$

6.1.4 Reflection



$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

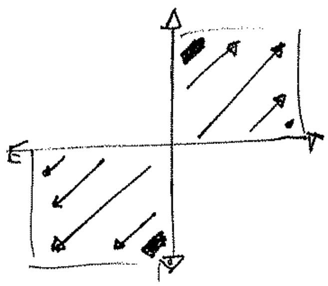
reflect about y-axis.

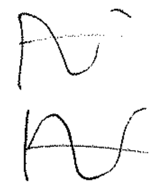


can't achieve without rigidly moving thoug[h] 3D.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

reflect about x-axis



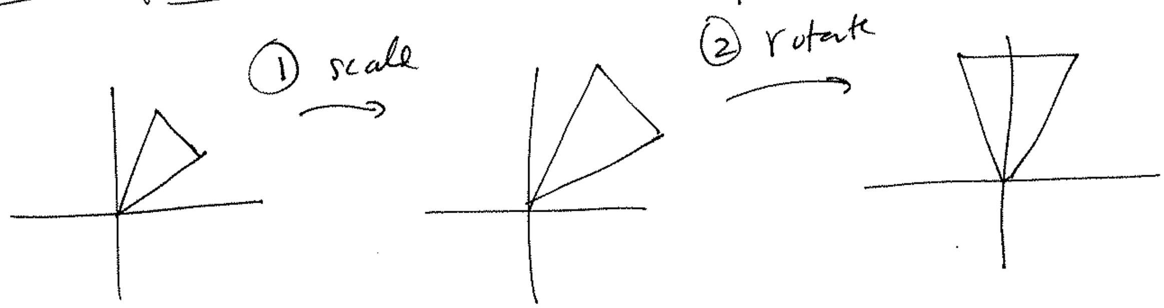


"reflect" about both

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = R(\pi)$$

6.1.5 Composition

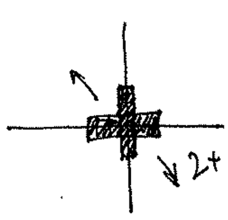


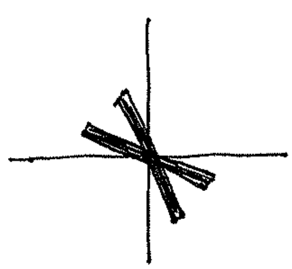
$$\underset{\sim}{v} = R S \underset{\sim}{u} = R(S\underset{\sim}{u}) = (RS)\underset{\sim}{u}$$

associativity

composition — Not Commutative !!!

Example : How to do this?





$$R(\tfrac{\pi}{4})S(1,2)R(-\tfrac{\pi}{4})$$

$$R^T S R$$

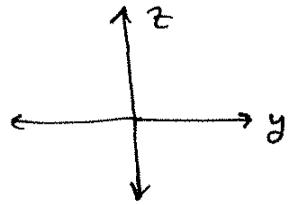
Q. what if we reversed order?

6.2 3D Transformation

$$scale(s_x, s_y, s_z) = \begin{pmatrix} s_x & & \\ & s_y & \\ & & s_z \end{pmatrix}$$

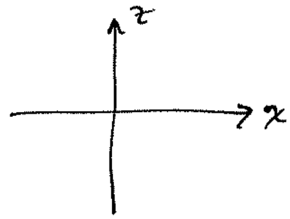
rotate

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$



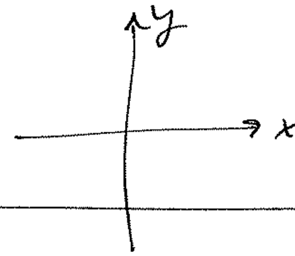
(x points out)

$$R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$



(y points into the board)

$$R_z(\phi) = \begin{pmatrix} \cos\phi & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



(z points out)

shear along x-axis

$$\begin{pmatrix} 1 & d_y & d_z \\ & 1 & \\ & & 1 \end{pmatrix}$$

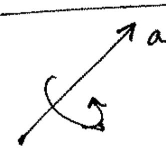
6.2.1 Arbitrary Rotations.

+ orthonormal rows :

$$R = \begin{pmatrix} - & u & - \\ - & v & - \\ - & w & - \end{pmatrix}$$

$$u \cdot u = v \cdot v = w \cdot w = 1$$
$$u \cdot v = u \cdot w = v \cdot w = 0$$

$$Ru = e_1$$
$$Rv = e_2$$
$$Rw = e_3$$



rotate about a by an angle θ.

$$R^T R_z(\theta) R$$

rotate a to z-axis.

6.3 Translation and Affine Transformation

linear transformation always maps $0 \to 0$. $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$x' = x + t_x$$

$$y' = y + t_y$$

→ can't do this w/ a 2×2 matrix.

→ use matrix + vector

→ or, use 3×3 matrix + homogeneous coordinates.

$$\underbrace{\begin{pmatrix} \boxed{\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}} & \boxed{\begin{matrix} t_x \\ t_y \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}} = \begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \\ 1 \end{pmatrix}.$$

"homogeneous coordinates"

similar in 3D

$$\begin{pmatrix} 1 & & & t_x \\ & 1 & & t_y \\ & & 1 & t_z \\ & & & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

translation in 3D. homogen. coords in 3D

* vectors that represent directions

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \leftarrow \text{don't get translated.}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

position direction

(*) Example: windowing transformation

For perspective viewing, homogeneous coord. will take on values other than 0, or 1.

$$\begin{pmatrix} 1 & & t_x \\ & 1 & t_y \\ & & 1 \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix}$$ transform, then translate

Note: $$\left(\begin{array}{cc|c} a & b & 0 \\ c & d & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} 1 & & t_x \\ & 1 & t_y \\ \hline & & 1 \end{array}\right) = \begin{pmatrix} a & b & t_x a + t_y b \\ c & d & t_x c + t_y d \\ 0 & 0 & 1 \end{pmatrix}$$ translate, then transform

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Rigid body transforms: translate + rotation

6.4 Inverses of Transformation Matrices

algebraic — just take matrix inverse

geometric — undo the transformations by doing opposite.

— equivalent

$$M = M_n \cdots M_2 M_1$$

$$M^{-1} = M_1^{-1} M_2^{-1} \cdots M_n^{-1}$$ note reversal of order.

- diagonal

$$\begin{pmatrix} s_1 & & \\ & s_2 & \\ & & s_3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s_1} & & \\ & \frac{1}{s_2} & \\ & & \frac{1}{s_3} \end{pmatrix}$$

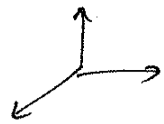
- rotation

$$R^{-1} = R^T$$

- matrix w/ $(0\ 0\ 0\ 1)$ in bottom row also has inverse w/ same bottom row

6.5 Coordinate Transformations

coordinate system has an

- origin $\underset{\sim}{p}$
- basis  $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$

coordinates (α, β, γ) describe a point in the frame

$$p + \alpha \underset{\sim}{u} + \beta \underset{\sim}{v} + \gamma \underset{\sim}{w}$$

$$p_{xyz} = \begin{pmatrix} u & v & w & p \\ 0 & 0 & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ 1 \end{pmatrix}}_{= p_{uvw}}$$

Q6.

$$\begin{pmatrix} R & t \\ 000 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^T & 0 \\ 000 & 1 \end{pmatrix} \begin{pmatrix} I & -t \\ 000 & 1 \end{pmatrix} = \begin{pmatrix} R^T & -R^T t \\ 000 & 1 \end{pmatrix}$$

<u>check:</u>

$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R^T & -R^T t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} RR^T & (-RR^T t + t) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$