general rotations

Rotation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

X axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Y axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Z axis

Rotation about an arbitrary axis $\mathbf{w}$ by θ radians

- **complete orthonormal basis**

$$\mathbf{u}, \mathbf{v}, \mathbf{w}$$

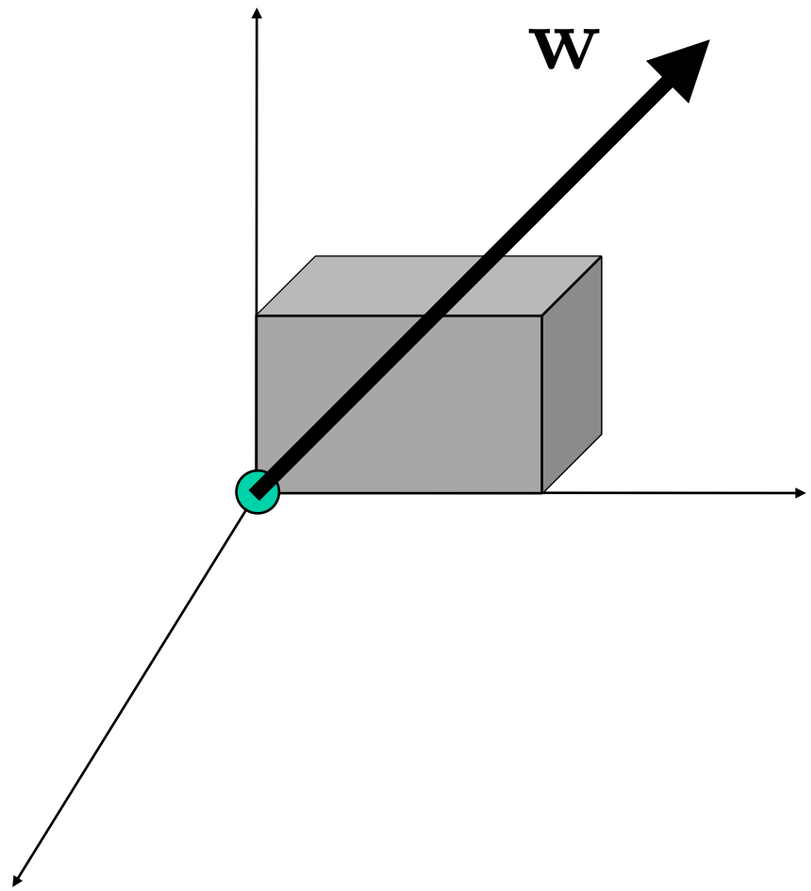
- **rotation matrix (w -> z-axis)**

$$R^T = \begin{pmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{pmatrix}$$

$$R = \begin{pmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{pmatrix}$$

- **composite rotation**

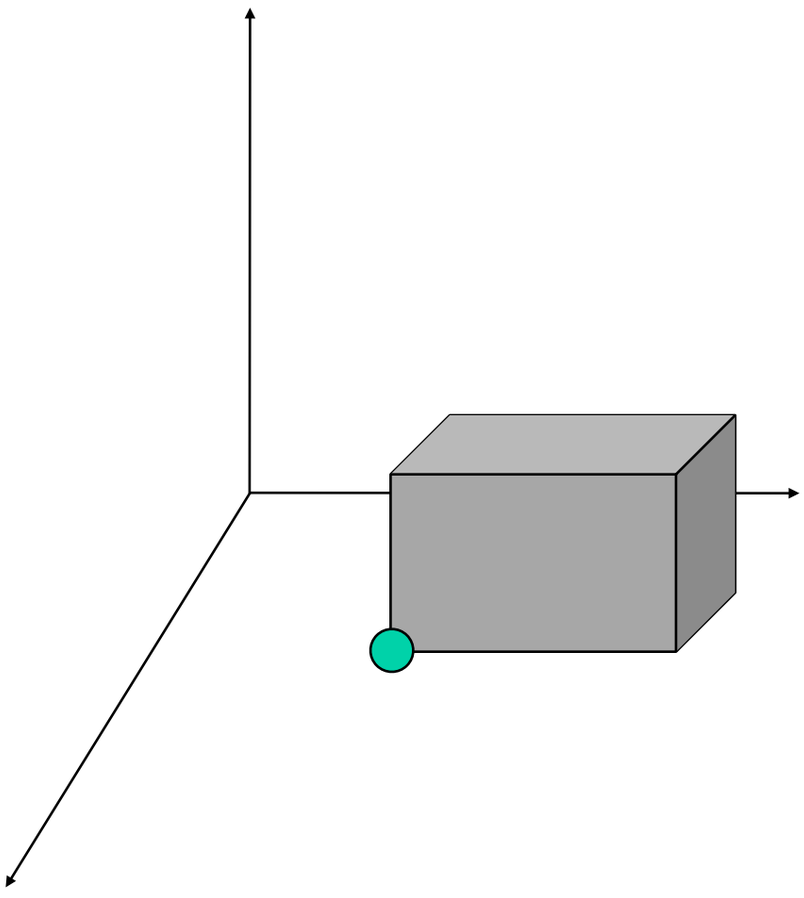
$$RR_z(\theta)R^T$$

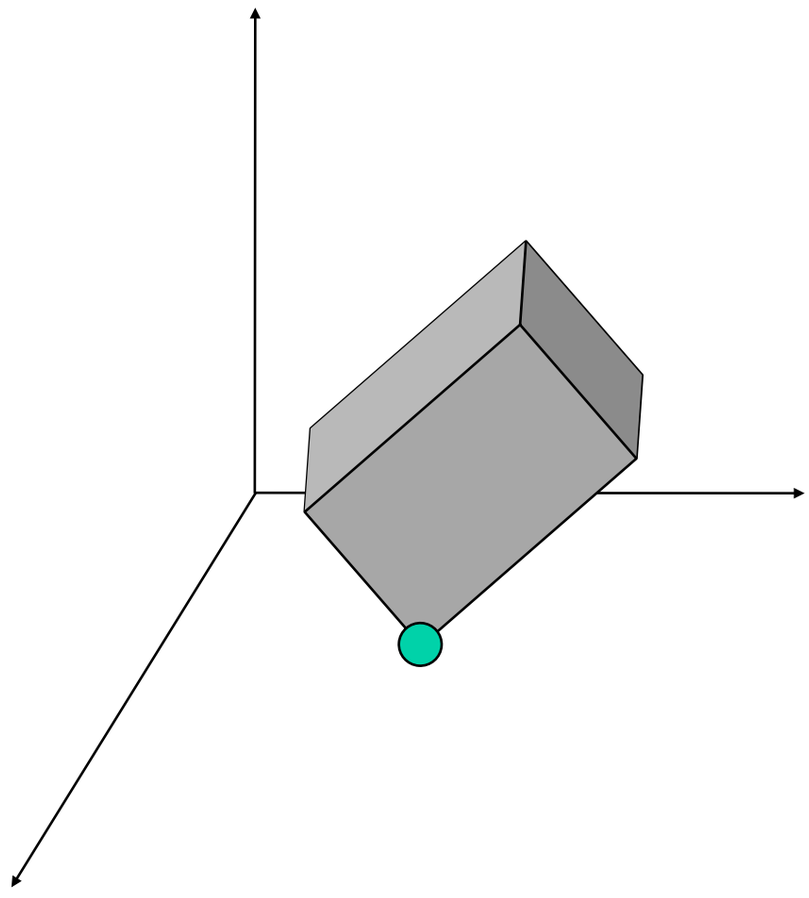


Composite Transformations

- Rotating about a fixed point
  - **basic** rotation alone will rotate about origin

but we want:

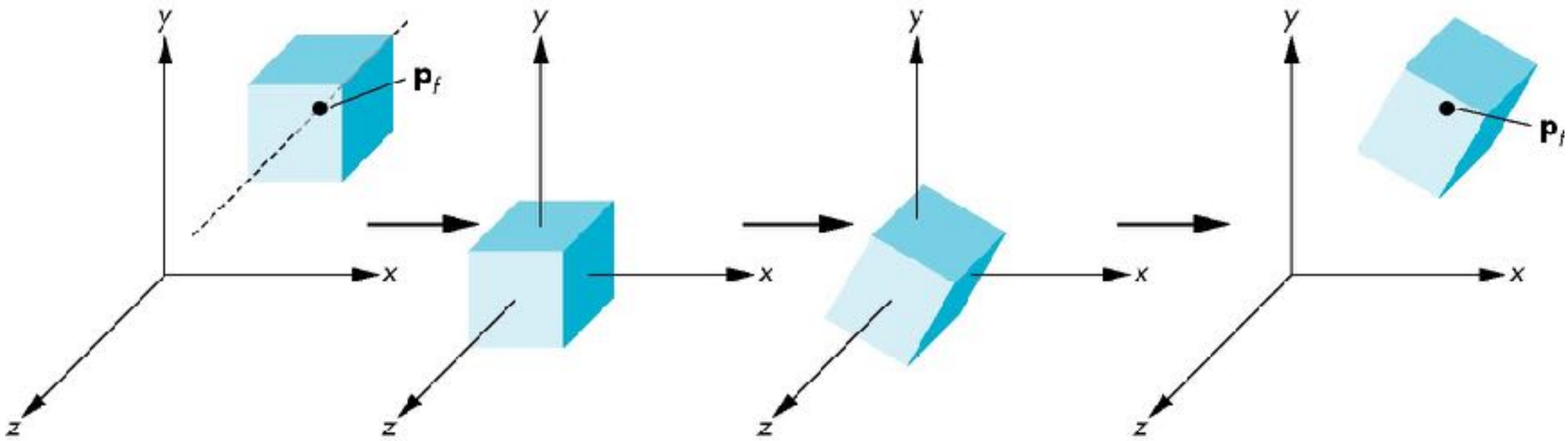




Composite Transformations

- Rotating about a fixed point
- Move fixed point (px,py,pz) to origin
- Rotate by desired amount
- Move fixed point back to original position

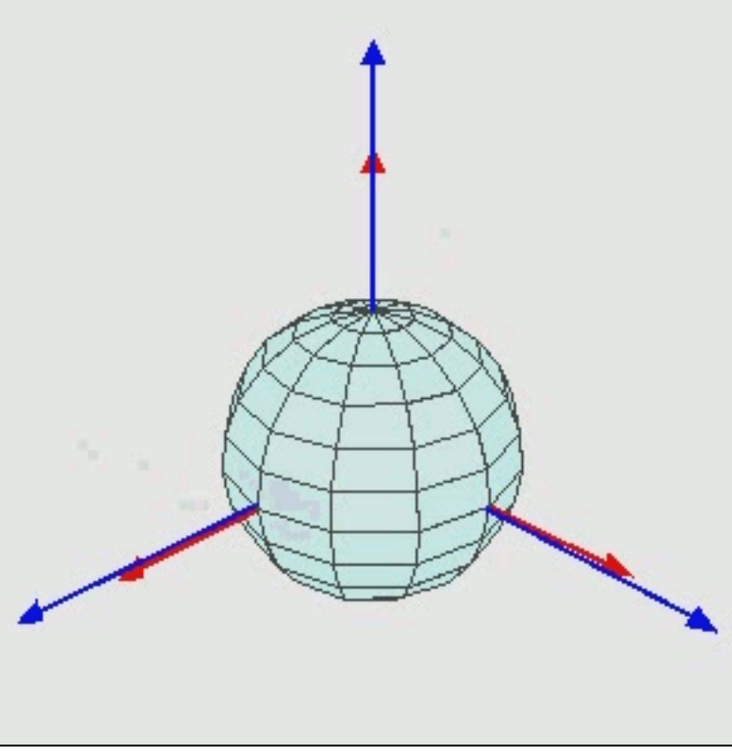
$$\mathbf{M} = \mathbf{T}(px, py, pz)\,\mathbf{R}\,\mathbf{T}(-px, -py, -pz)$$

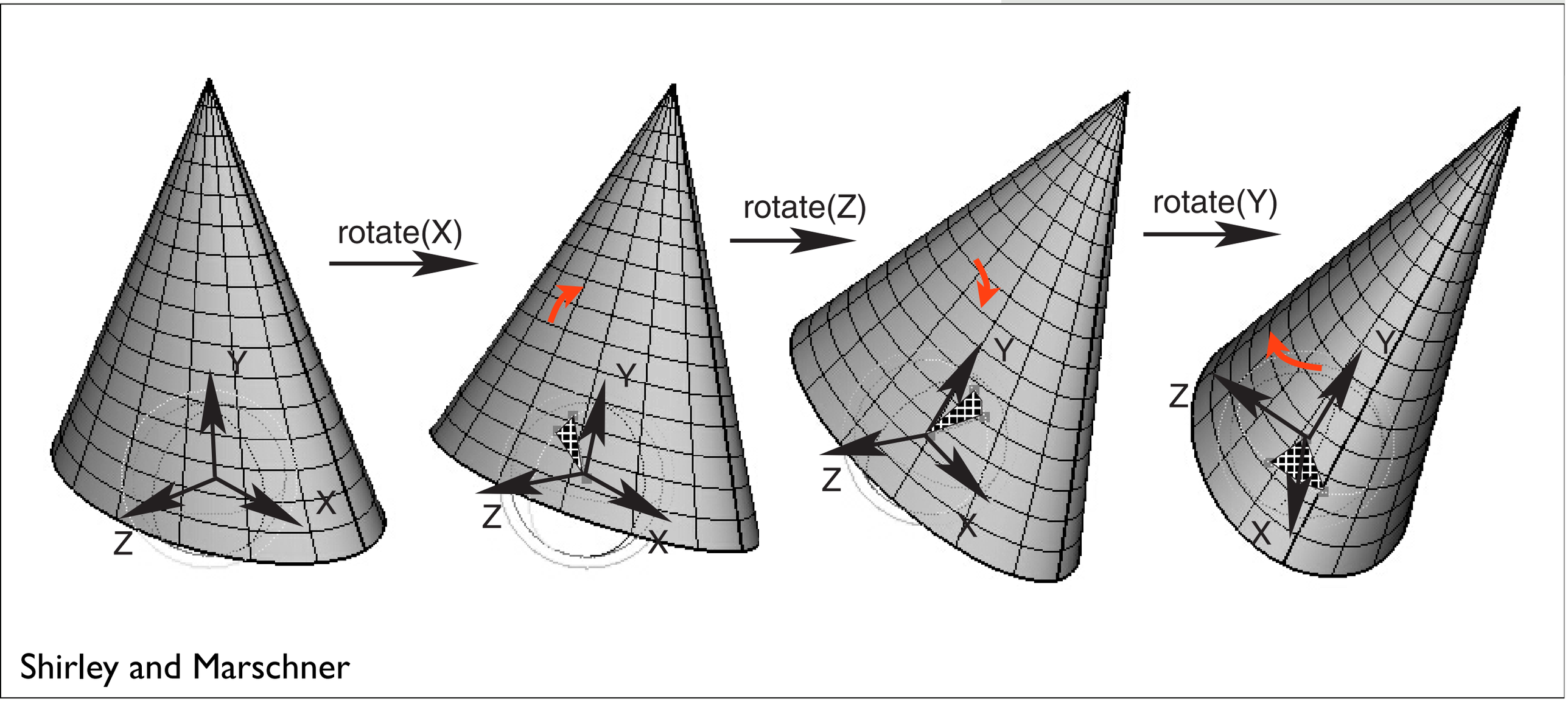


Euler angles

Euler Angles

3 Euler angles can be used to specify an arbitrary orientation

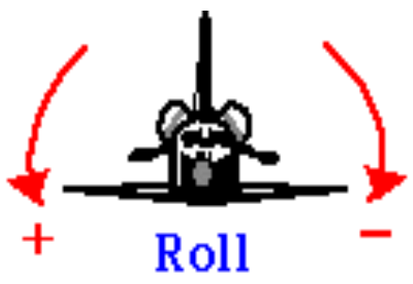


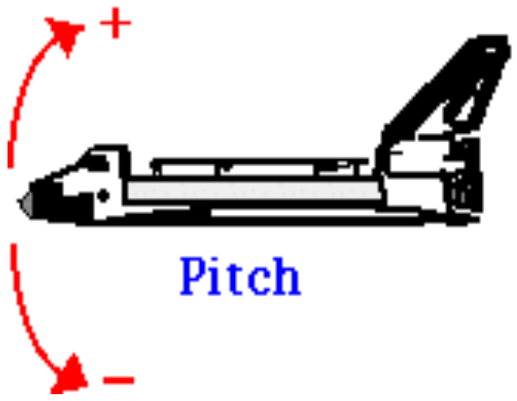


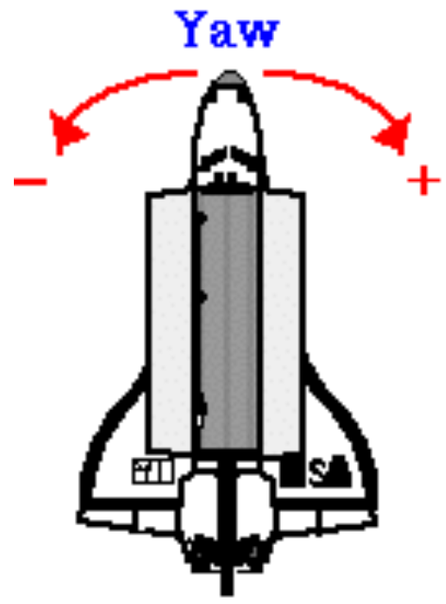
Shirley and Marschner

Euler Angles

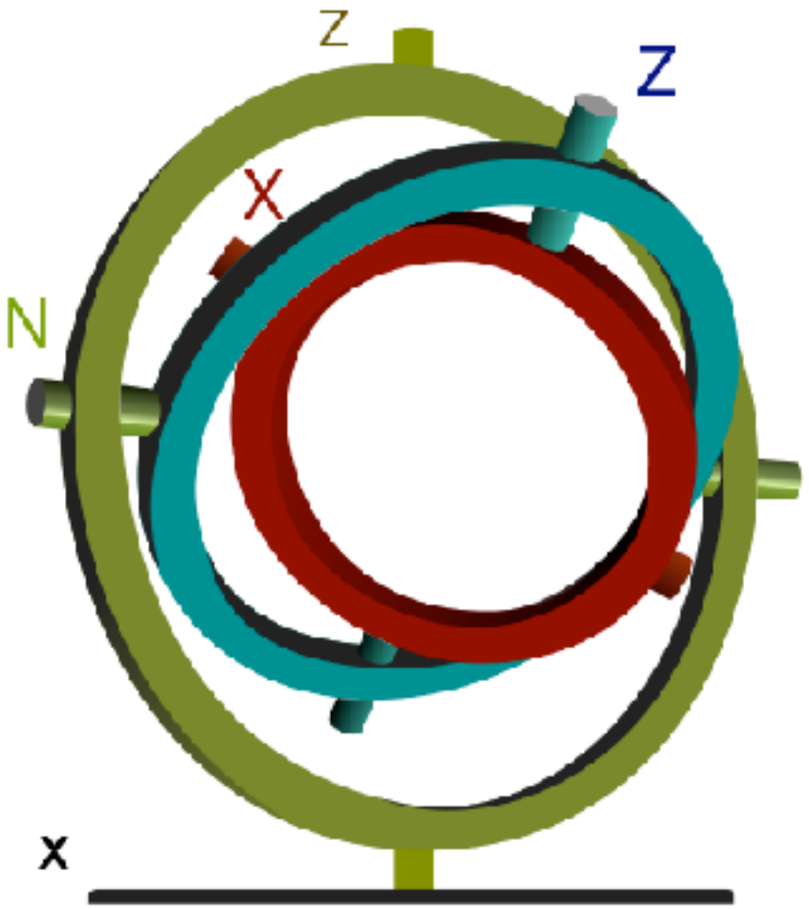
- A general rotation is a combination of three elementary rotations:
  - around the x-axis (roll)
  - around the y-axis (pitch)
  - around the z-axis (yaw)

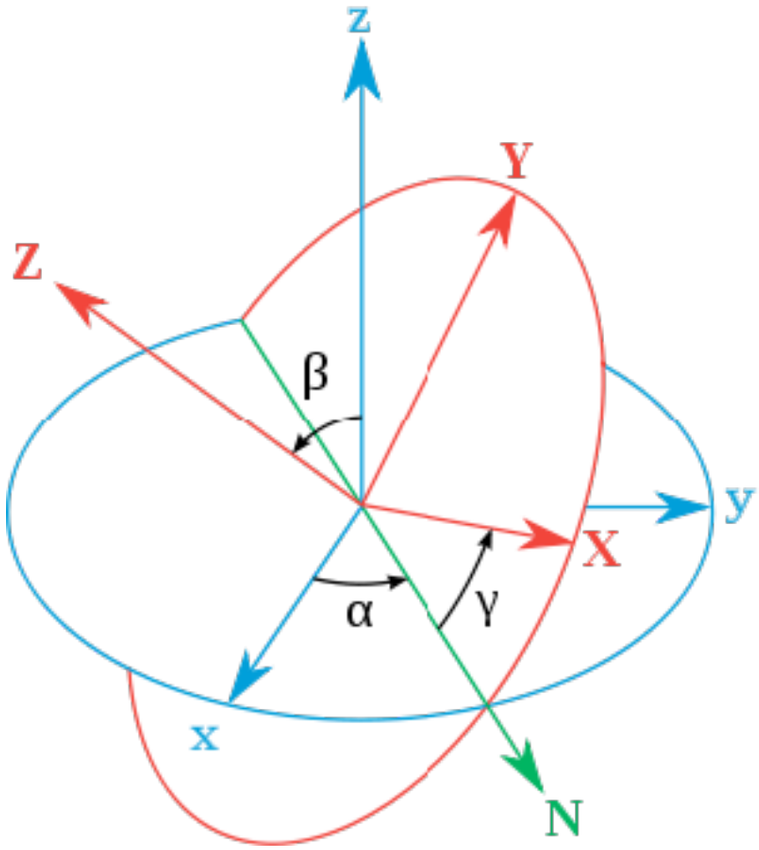






Gimbal and Euler Angles





Z-X'-Z"

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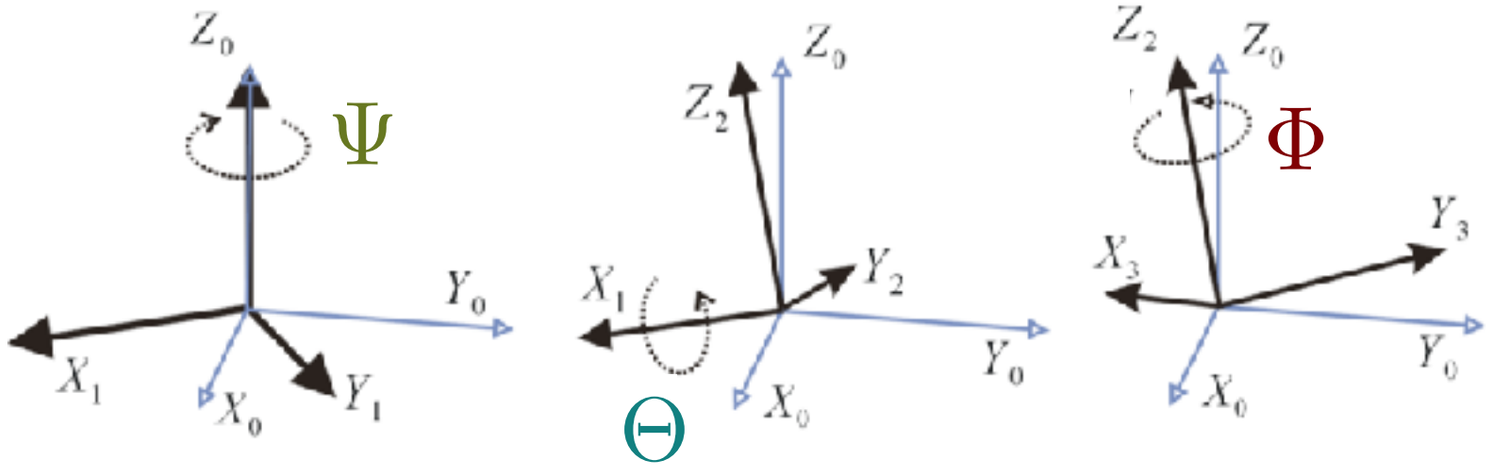
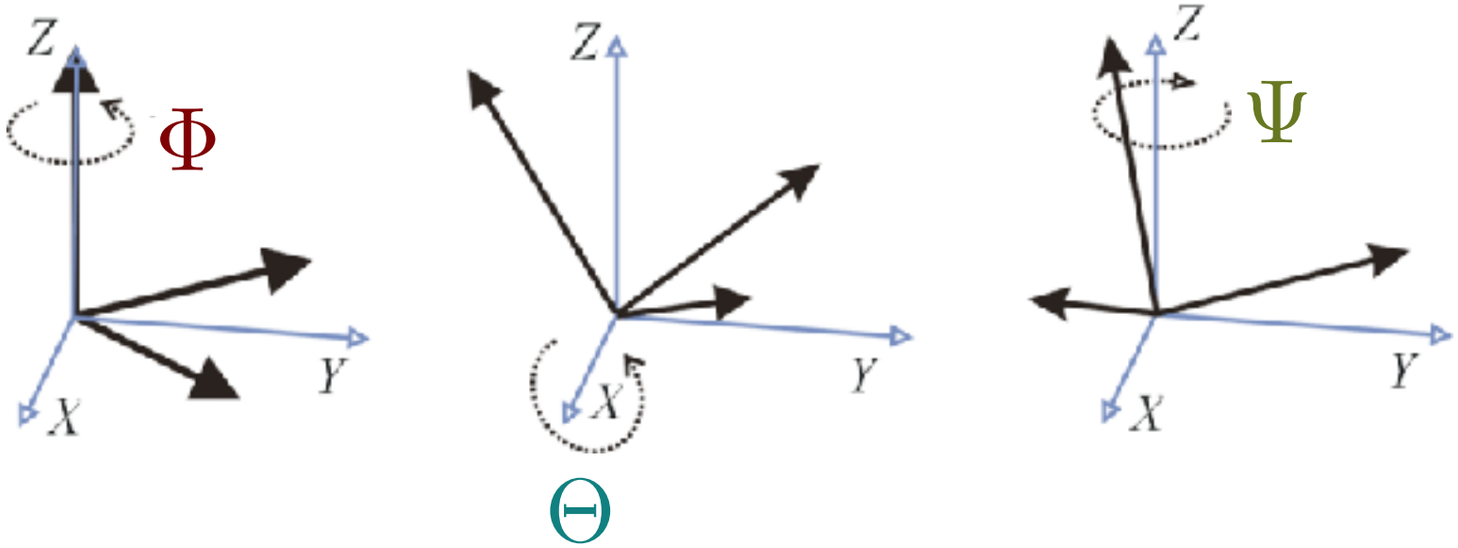
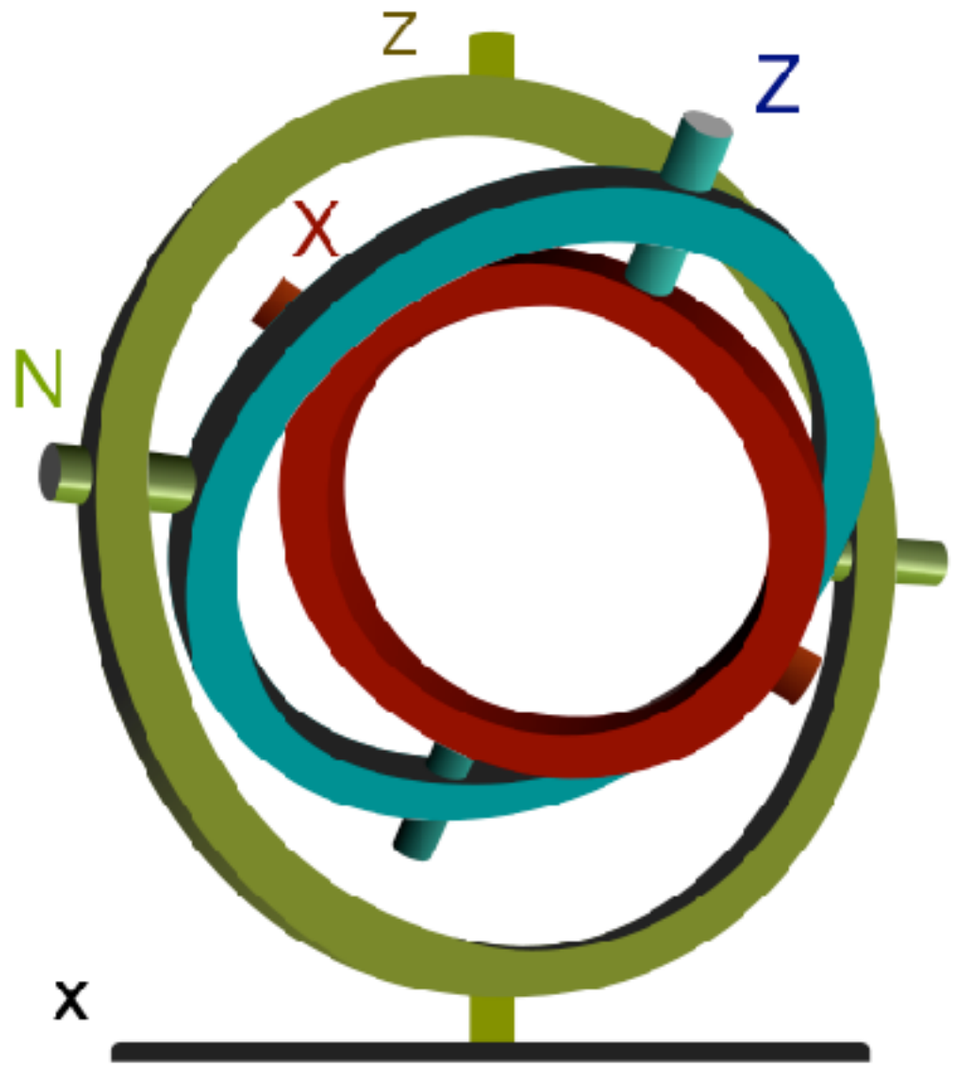
intrinsic



extrinsic





[Wikimedia Commons]

Euler Angles and Rotation Matrices

$$\text{x-roll}(\theta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{y-pitch}(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{z-yaw}(\theta_3) = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{R}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} c_2c_3 & -c_2s_3 & s_2 & 0 \\ s_1s_2c_3 + c_1s_3 & -s_1s_2s_3 + c_1c_3 & -s_1c_2 & 0 \\ -c1s_2c_3 + s_1s_3 & c_1s_2s_3 + s_1c_3 & c_1c_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

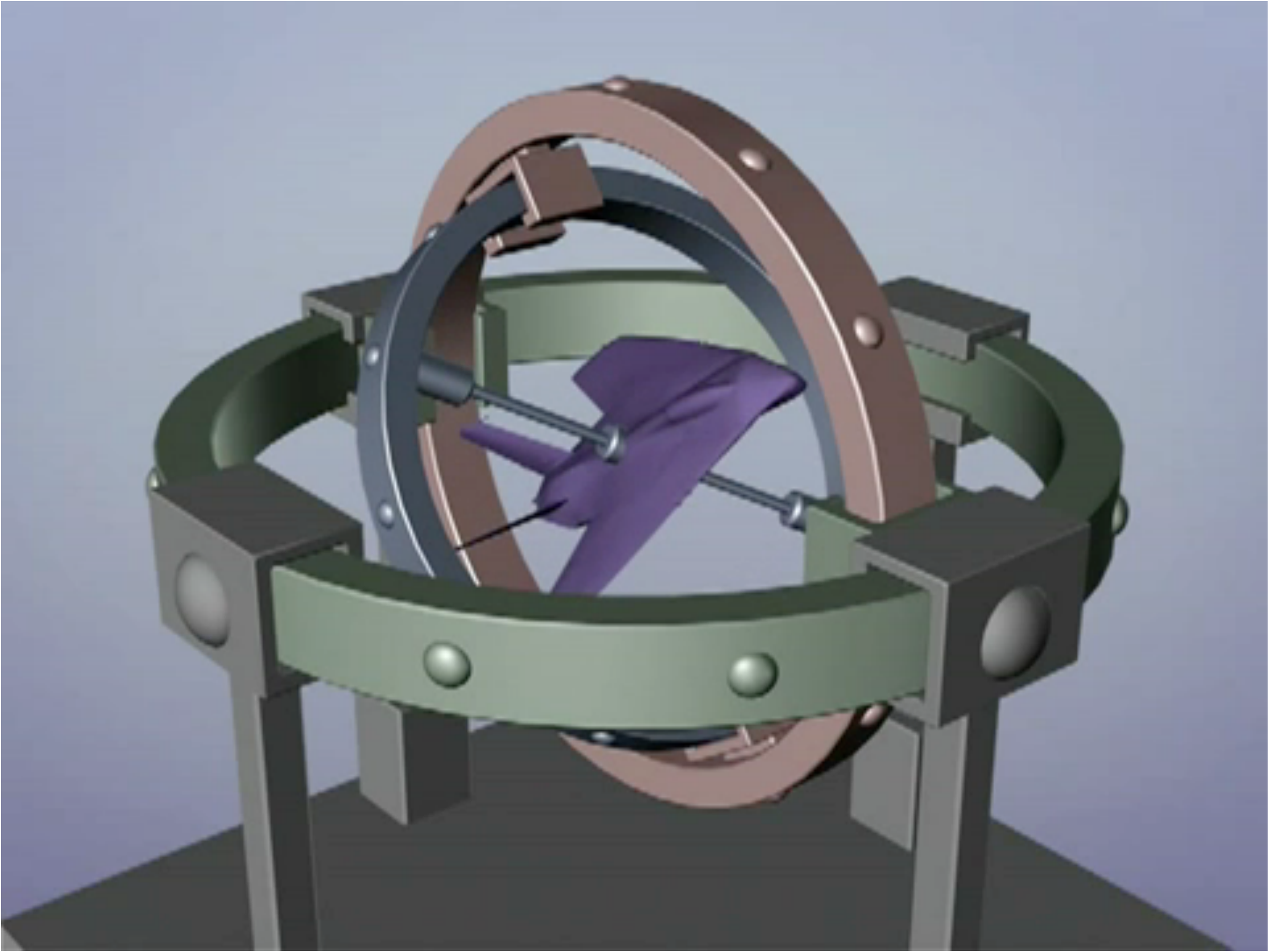
Interpolating Rotations

- naive element-by-element interpolation of rotation matrices does not work! E.g.,

$$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

90 CW 90 CCW

- We can interpolate Euler angles, but there is a problem known as Gimbal Lock



quaternions

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication

$$i^2 = j^2 = k^2 = ijk = -1$$

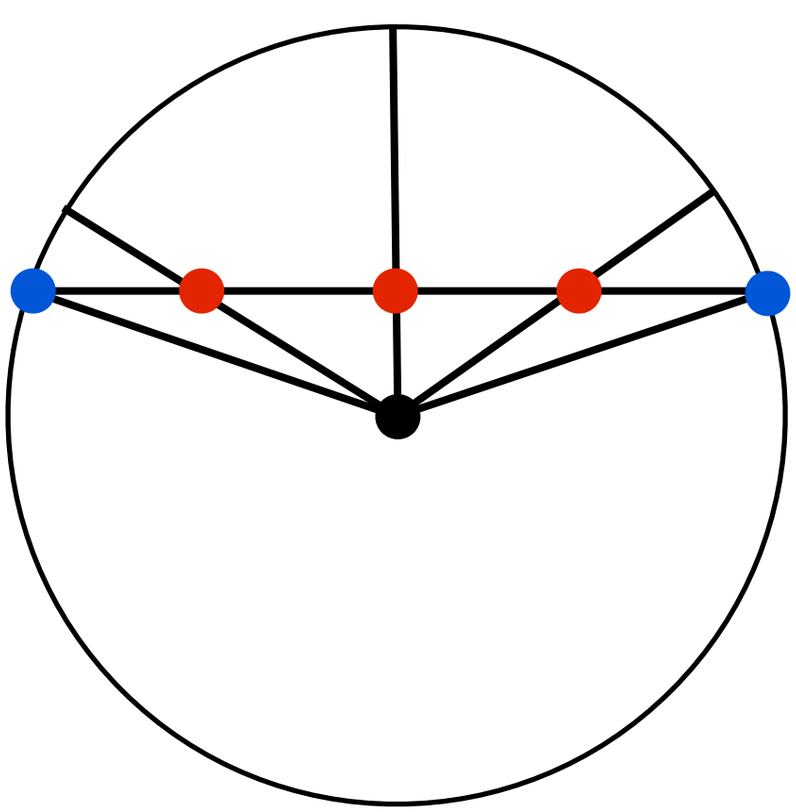
& cut it on a stone of this bridge

Quaternions

- axis/angle representation
- interpolates smoothly
- easy to compose

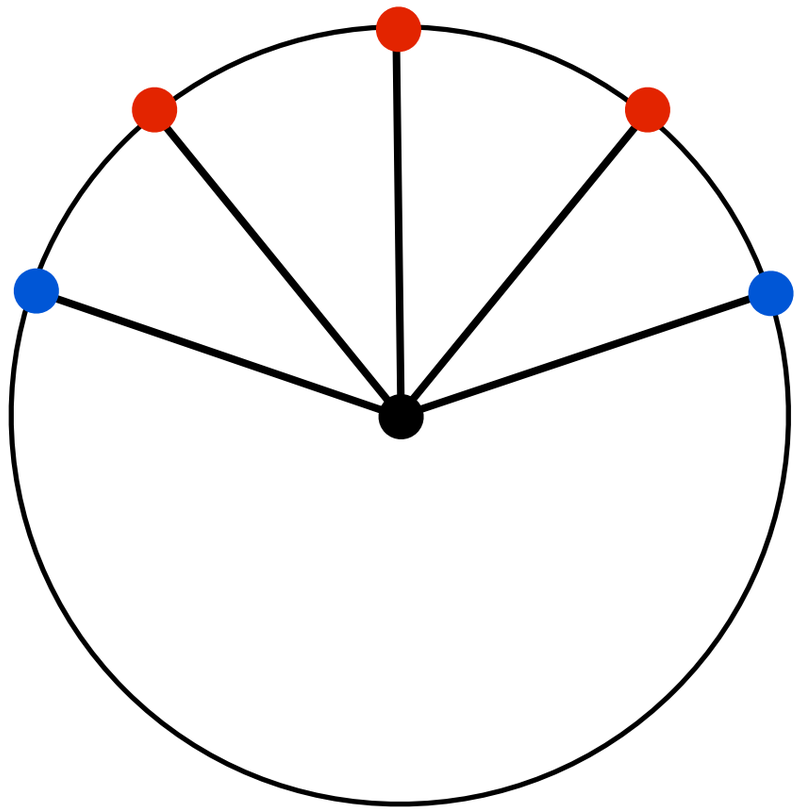
<whiteboard>

Quaternion Interpolation



linear

non-uniform speed



spherical linear (slerp)

constant speed

Matrix form

$$\mathbf{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\mathbf{R}(\mathbf{q}) = \begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy + 2wz & 2xz - 2wy & 0 \\ 2xy - 2wz & 1 - 2x^2 - 2z^2 & 2yz + 2wx & 0 \\ 2xz + 2wy & 2yz - 2wx & 1 - 2x^2 - 2y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotations in Reality

- It's easiest to express rotations in Euler angles or Axis/angle

- We can convert to/from any of these representations

- Choose the best representation for the task
  - input:Euler angles
  - interpolation: quaternions
  - composing rotations: quaternions, orientation matrix