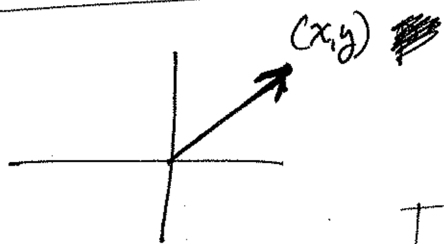
LECTURE 7

6.1.1 - 6.1.5
6.2.0 - 6.2.1
6.3
- 6.5

geometric transformations:
- rotation
- shear
- translation
- scaling
- projection

} can be accomplished with transformation matrices

6.1 2D linear transformation



a point represented as an offset from the origin.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

2D linear transformation

(*) Linear
$A(\underset{\sim}{u} + \underset{\sim}{v}) = Au + Av$
$A(\alpha \underset{\sim}{u}) = \alpha A \underset{\sim}{u}$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

A maps one 2D vector to another.

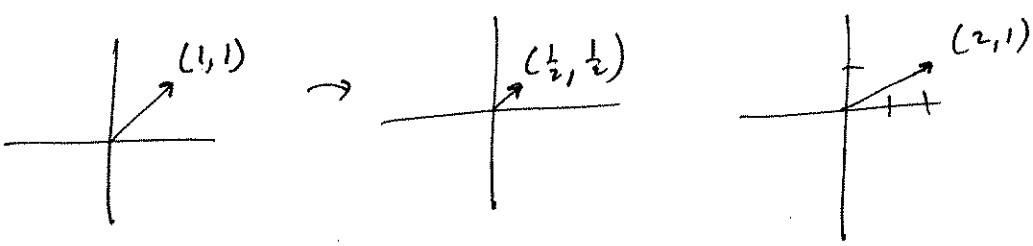
Choice of A entries yield diff. transformations.

6.1.1 Scaling

$$\begin{pmatrix} S_x & 0 \\ 0 & S_y \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_x x \\ S_y y \end{pmatrix}$$

examples of uniform & non uniform

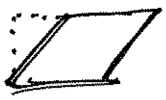
examples



6.1.2 Shearing.

"deck of cards"

horizontal shear $\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$

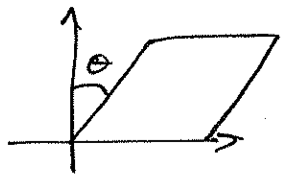


vertical shear

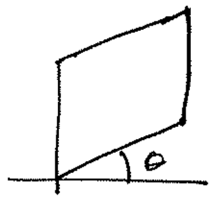
$\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ 45°

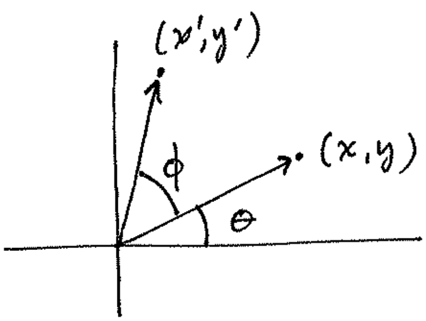
Generally, $\begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$



$\begin{pmatrix} 1 & 0 \\ \tan\theta & 1 \end{pmatrix}$



6.1.3 Rotation.



rotates counterclockwise

$$(x, y) = (\cos\theta, \sin\theta)$$

$$(x', y')^T = \begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{pmatrix} = \begin{matrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \cos\theta\sin\phi + \cos\theta\sin\phi \end{matrix}$$

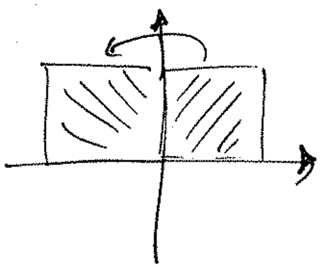
$$= \begin{pmatrix} x\cos\phi - y\sin\phi \\ x\sin\phi + y\cos\phi \end{pmatrix} =$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}}_{\substack{\text{rotate}(\phi) \\ R(\phi)}} \begin{pmatrix} x \\ y \end{pmatrix}$$

to rotate clockwise by ϕ, rotate c.c.w. by $-\phi$

(*) Rotation matrix is an orthogonal matrix. $R(\phi)^T R(\phi) = R(\phi) R(\phi)^T = I$.

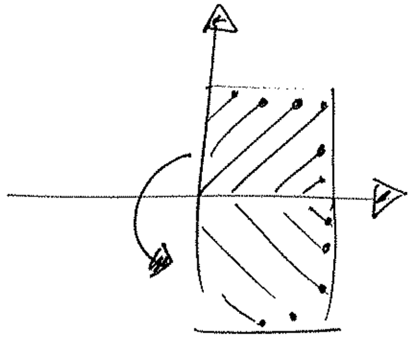
6.1.4 Reflection



$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

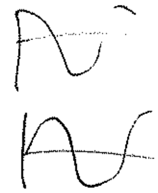
reflect about y-axis.

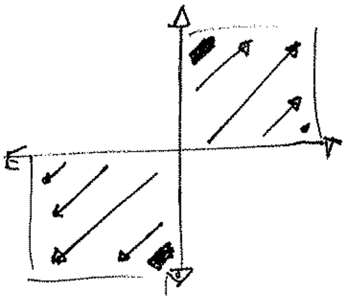
can't achieve without rigidly moving through 3D.



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

reflect about x-axis



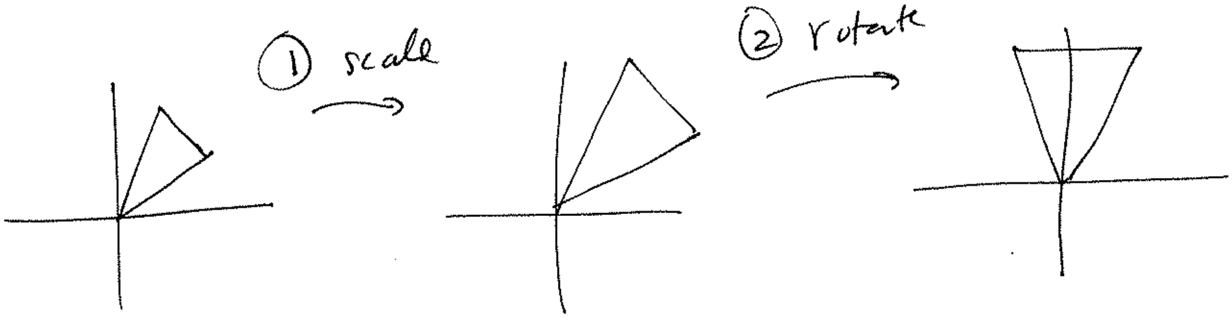


"reflect" about both

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix} = R(\pi)$$

6.1.5 Composition



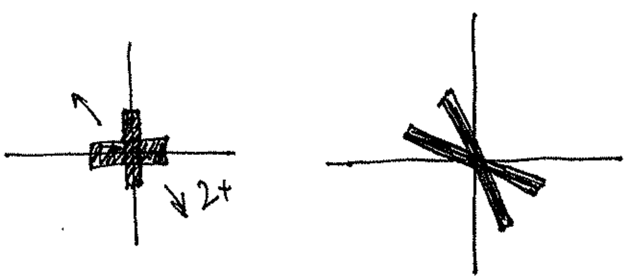
$$\underset{\sim}{v} = R S \underset{\sim}{u} = R(S\underset{\sim}{u}) = (RS)\underset{\sim}{u}$$

associativity

composition — <u>Not Commutative</u> !!!

<u>Example</u> : How to do this?

$$R\left(\tfrac{\pi}{4}\right) S(1,2)\, R\left(-\tfrac{\pi}{4}\right)$$



$$R^T S R$$

Q. what if we reversed order?

6.3 Translation and Affine Transformation

linear transformation always maps $0 \to 0$. $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$x' = x + t_x$$

$$y' = y + t_y$$

→ can't do this w/ a 2×2 matrix.

→ use matrix + vector

→ or, use 3×3 matrix + homogeneous coordinates.

$$\underbrace{\begin{pmatrix} \boxed{\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}} & \boxed{\begin{matrix} t_x \\ t_y \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}} = \begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \\ 1 \end{pmatrix}.$$

↑ "homogeneous coordinates"

Similar in 3D

$$\begin{pmatrix} 1 & & & t_x \\ & 1 & & t_y \\ & & 1 & t_z \\ & & & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

translation in 3D. homogen. coords in 3D

* vectors that represent directions

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \leftarrow \text{don't get translated.}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

position   direction

(*) Example: windowing transformation

For perspective viewing, homogeneous coord will take on values other than 0, or 1.