

Lecture 3 Notes

- Math Review

1. points — locations $\cdot P$

$\cdot Q$

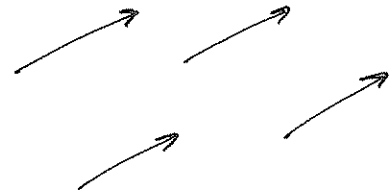
P, Q, R

2. Vectors - direction & magnitude

$\vec{u}, \vec{v}, \vec{w}$

- no notion of location

- all relative



• vector addition

(+, -) • scalar multiplication

scalars: α, β, γ

• vector space

- coordinate system & basis vectors.

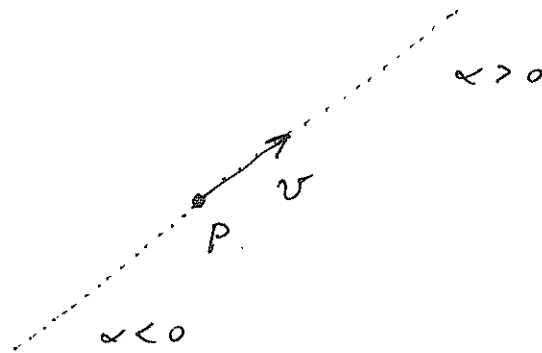
$$\vec{a} = (a_1, a_2, a_3) = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

3. Vector space vs. affine space

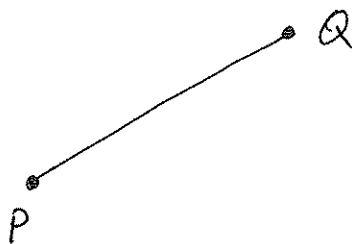
↳ point + vector

4. Lines

$$P(\alpha) = P + \alpha \vec{v}$$



line segments



$$(1-\alpha)P + \alpha Q$$

$$0 \leq \alpha \leq 1$$

equivalent:

$$P + \underbrace{\alpha(Q-P)}_{=\vec{v}}$$

5. Dot Product

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$$

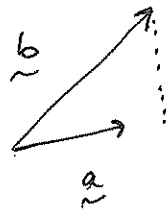
$$\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

1x3 3x1

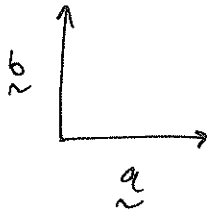
$$\vec{a} \cdot \vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3 = \|\vec{a}\|^2$$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

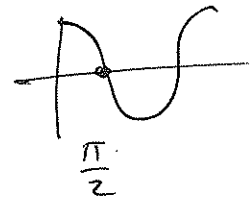
geometric interpretation: (\vec{a} unit vector)



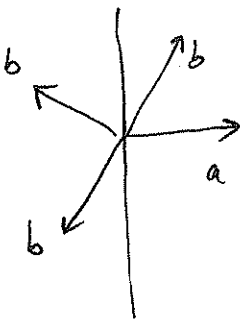
Q



$$\vec{a} \cdot \vec{b} = ?$$



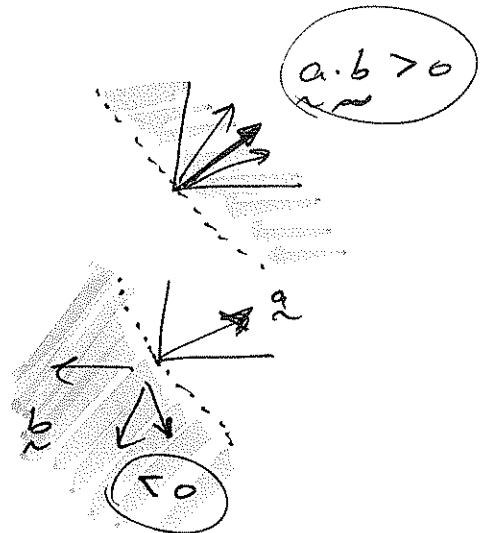
Q



$$\vec{a} \cdot \vec{b} > 0$$

$$=$$

$$< 0$$



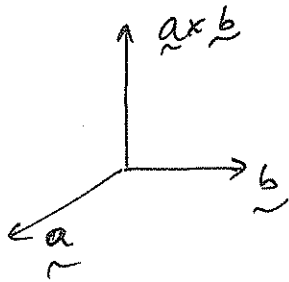
region of
-ive
dot product

Cross Product

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2) \hat{i} + (a_3 b_1 - a_1 b_3) \hat{j} + (a_1 b_2 - a_2 b_1) \hat{k}$$

result of cross product is another vector!

Right-hand rule:



$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

magnitude of the resulting vector

Q $\vec{a} \times \vec{a} = ?$

$= 0$

direction is given by right-hand rule.

Points are locations

P

Q

use uppercase for points

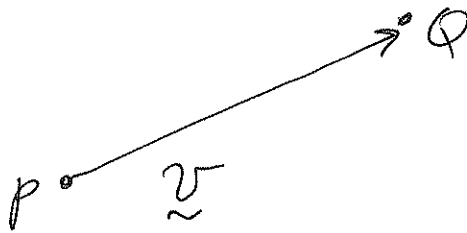
Vectors

- Cartesian "canonical" basis 3D.

\hat{i} , \hat{j} , \hat{k}

- other bases can be used

$$\vec{v} = Q - P$$



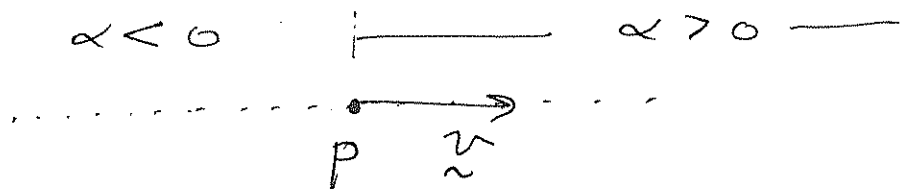
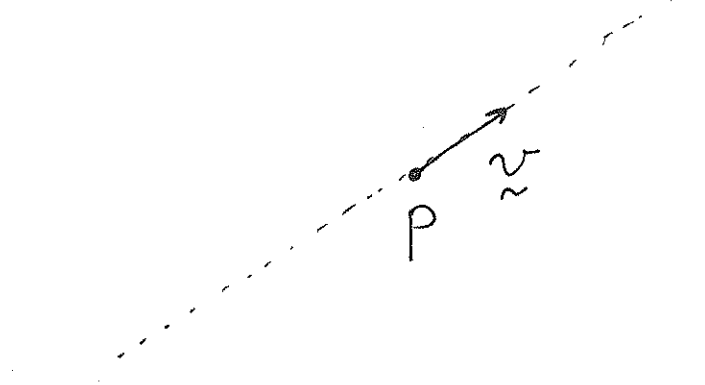
$$\underline{Q} = P + \underline{\vec{v}}$$

line is 1D

because it can be parametrized
by 1 number.

↳ α

$$p(\alpha) = P + \alpha \vec{v}$$



$$f_1(x) = Ax$$

$$f_2(x) = Bx$$

matrix mult.
as a sequence
of transformations

$$f(x) = f_1(f_2(x)) = A(f_2(x)) \\ = \underbrace{(A \ B)} x$$

$$C = AB$$

$$f(x) = (AB)x$$

$$f(x) = Cx$$

$$f_1 \circ f_2$$

transpose of a matrix

$$a_{ji} \longleftarrow a_{ij}$$

matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

a_{ij}
 i^{th} row
 j^{th} column

2 rows $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$
3 columns

2x3 matrix

matrix multiplication

$$A \quad B$$

$m \times k$ $k \times n$

- you can't just multiply any two matrices
- they have to be Compatible.

$$y = Ax$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} x_2 + \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} x_3$$

$$f(x)$$

linear if + only if

$$\textcircled{1} \quad f(\alpha x) = \alpha f(x)$$

$$\textcircled{2} \quad f(x+y) = f(x) + f(y)$$

$$f(x) = \underbrace{\alpha}_{\text{vector}} (mx) + \underbrace{b}_{\text{point}} \quad \text{AFFINE}$$

$$\textcircled{1} \quad f(\alpha x) = m\alpha x + b = \alpha \left(mx + \frac{b}{\alpha} \right) \neq \alpha f(x)$$

$$\textcircled{2} \quad f(x+y) = m(x+y) + b = mx + my + b \neq f(x) + f(y)$$

"linear in x "

"linear" (highest order term is linear).

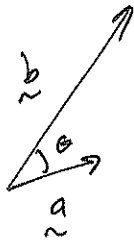
$$\underline{b = 0}$$
$$f(x) = mx$$

LINEAR

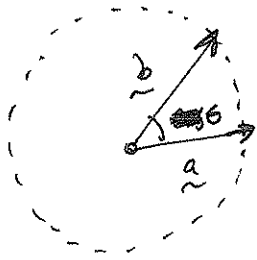
$$\textcircled{1} \quad f(\alpha x) = m\alpha x = \alpha (mx) = \alpha f(x) \checkmark$$

$$\textcircled{2} \quad f(x+y) = m(x+y) = mx + my \checkmark$$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

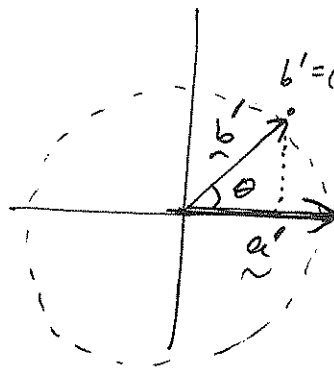


① assume \vec{a}, \vec{b} are unit vectors



$$\vec{a} \cdot \vec{b} = \cos \theta$$

rotate \vec{a} & \vec{b}
so \vec{a} is
on x-axis

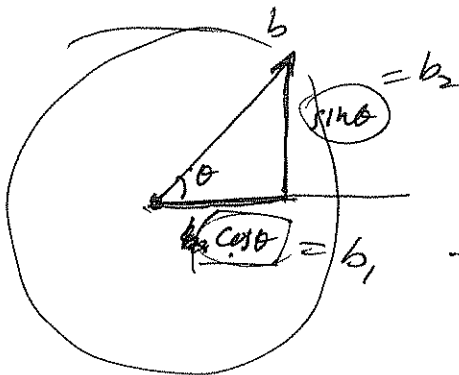


$$\vec{a}' = (1, 0)$$

$$\vec{b}' = (b'_1, b'_2)$$

$$\vec{a} \cdot \vec{b} = b'_1 = \cos \theta$$

→ haven't changed θ !



R rotation

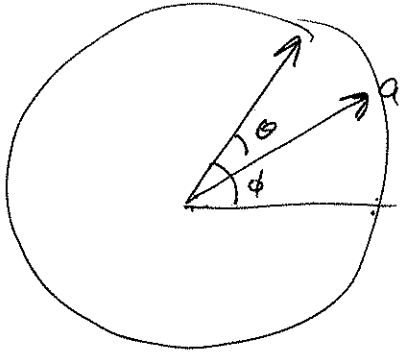
$$\vec{a}' = R \vec{a}$$

$$\vec{b}' = R \vec{b}$$

$$(\vec{R} \vec{a}) \cdot (\vec{R} \vec{b}) = (\vec{R} \vec{a})^T (\vec{R} \vec{b})$$

$$= \vec{a}^T \vec{R}^T \vec{R} \vec{b} = \vec{a}^T \vec{b}$$

$$b = (\cos \phi, \sin \phi)$$



→
rotate

